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Warp Drive Spacetimes

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WARP DRIVE SPACETIMES

A thesis submitted in partial fulfillment of the requirements for the degree of Master
of Science at Virginia Commonwealth University.

by

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The concept of faster than light travel in general relativity is examined, starting with a review of the Alcubierre metric. This spacetime, although incredible in its implications, has certain unavoidable problems which are discussed in detail. It is demonstrated that in order to describe faster than light travel without any ambiguities, a coordinate independent description is much more convenient. An alternative method of describing superluminal travel is then proposed, which has similarities to the Krasnikov tube.

CHAPTER 1

INTRODUCTION

One of the greatest triumphs in theoretical physics is the theory of relativity. Without relativistic corrections, our Global Positioning System (GPS) would be useless. General relativity has also been used to calculate, with incredible success, the motion of planets, stars, and satellites. In fact, one of the first major successes of general relativity was finally explaining the observed perihelion precession of Mercury, which was not accurately represented in Newtonian gravity. Recently, the LIGO inteferometric observatories confirmed another prediction of general relativity: gravitational waves[1–6]. The findings of LIGO match numerical relativistic predictions with uncanny accuracy. Additional experimentally confirmed predictions of general relativity include gravitational lensing and and the gravitational redshift. To date, the theory has stood up to almost every experimental challenge, with the only notable exception being at very large distances in the case of galaxy rotation curves.

When Albert Einstein invented special relativity, he came up with one of the most famous rules in physics: objects with mass can approach, but never reach or exceed, the speed of light. About ten years later, Einstein implicitly came up with a way to break his own rule in the form of general relativity.

When moving from the framework of special relativity to that of general relativity the mathematics certainly become more complicated, but the rules appear to be more relaxed. Since spacetime is curved by the presence of mass, we can no longer make comparisons of velocity unless they are limited to very small distance scales, over which the rules of special relativity "kick back in". Thus, the cosmic speed limit need

only be obeyed locally. This fact, combined with the compelling predictive power demonstrated by general relativity thus far might lead one to ask the question "Is it truly possible to exceed the speed of light?"

It was this concept which led Miguel Alcubierre to come up with the first mathematical description of superluminal (faster than light) travel in the form of the "warp drive" metric. His solution to the Einstein field equations permits for superluminal travel by the apparent mechanism of distorting spacetime in a particular region, the "warp bubble" in such a way that the cosmic speed limit can be broken. This prescription for space travel is not without its problems though; and so we seek to describe the warp drive in a mathematically rigorous manner, and analyze the implications of the Alcubierre spacetime and related metrics through various lenses of physics.

Although warp drive spactimes themselves may have unphysical qualities which could exclude the possibility of ever experimentally realizing a warp bubble, they can be used quite effectively to probe the foundations and limits of general relativity, for the purpose of better understanding the underpinnings of the theory.

We begin with a review of the primary mathematical language used for general relativity: differential geometry. This will by no means be a thorough review of differential geometry, as it will be skewed toward the purpose of taking the fastest route to the fundamental equations of general relativity: the Einstein field equations. Next, special topics in the theory will be presented which are necessary for the understanding of warp drive spacetimes and their implications, after which we move to a review of the literature surrounding warp drives from past to present. Finally, an alternative methed of characterizing superluminal travel using a light cone based coordinate system is presented.

CHAPTER 2

REVIEW OF GENERAL RELATIVITY

Before continuing the discussion of Warp drives, the necessary mathematical and physical frameworks must be laid down. This section utilizes the textbooks [7–10] for reference.

In general relativity, gravity is not actually modeled as a force between objects, as it is in Newtonian mechanics. Gravity is instead thought of as a manifestation of the curvature of spacetime, which is brought about by the presence of matter and energy density. The path that a particle takes in the presence of a gravitational field is a geodesic, or the equivalent of a "straight line" in a curved space. In the next several sections, the key concepts of general relativity are reviewed, along with some specialized topics for the purpose of describing warp drive spacetimes.

2.1 Manifolds and Maps

In order to model the curvature of spacetime and describe gravitation the way Einstein intended, we must decide what exact mathematical object to take as our representation of spacetime itself. An important property we want this structure to have is that over sufficiently small distances spacetime is locally Minkowskian, or "flat", but over larger distances the curvature can still be nontrivial. The object satisfying these properties is called a **manifold**.

One can rationalize an intuitive definition of manifolds simply by examples: the surface of a sphere or a torus, the Euclidian space $\mathbf{R}^n$, the surface of a table, or even a single cone. In general, a manifold can be described as a space that may be curved

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Fig. 2.1.1.: A composite map. In this composition the map ϕ first acts on $a \in A$ to give the **image** $\phi(a) \in B$. Next the map ψ acts on $\phi(a)$ to give $\psi(\phi(a)) \in C$.
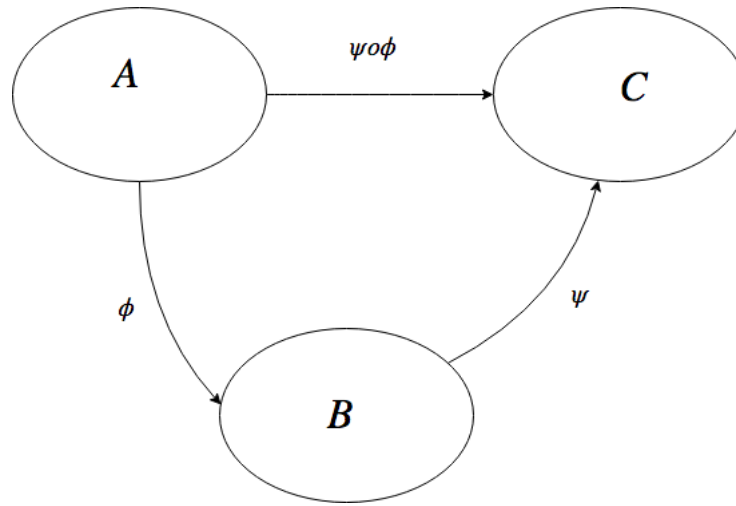
and have a complicated topology, but locally looks just like $\mathbf{R}^n$. There are some subtleties that become obscured by this definition, however, and so a more formal treatment is necessary.

We begin with the notion of a **map** between two sets. For two sets M and N, a map $\phi : M \to N$ is an object which assigns exactly one element of N to each element of M. A map generalizes the concept of a function. Additionally, given two maps $\phi : A \to B$ and $\psi : B \to C$ we can define the **composition** of ϕ and ψ as $\psi \circ \phi : A \to C$. In more familiar notation we can write the composition as $(\psi \circ \phi)(a) = \psi(\phi(a))$. In this composition the map ϕ first acts on $a \in A$ to give the **image** $\phi(a) \in B$. Next the map ψ acts on $\phi(a)$ to give $\psi(\phi(a)) \in C$. This relationship is represented graphically in Figure 2.1.1.

A map is **one-to-one** if each element of N has at most one element of M mapped into it, and **onto** if each element of N has at least one element of M mapped into it. A map which is both one-to-one and onto is called **invertible** or **bijective**. In that case we can define the **inverse map** $\phi^{-1} : N \to M$. In the more familiar notation this becomes $(\phi^{-1} \circ \phi)(a) = \phi^{-1}(\phi(a)) = a$.

Now we take a brief intermission to define and clarify the concept of **differentiability**. We seek a fundamental definition of a derivative, which will apply to all vector spaces we will discuss in the future. Begin with a map $K : X \to V$, where X and V are both vector spaces. Consider two vectors $\mathbf{x} \in X$ and $\mathbf{v} \in X$. The **Fréchet derivative** with respect to a real parameter ϵ can then be defined

$$D_{\mathbf{x}}K(\mathbf{v}) \equiv \frac{d}{d\epsilon}K(\mathbf{v} + \epsilon\mathbf{x})\Big|_{\epsilon=0}. \tag{2.1.1}$$

The Fréchet derivative can be generalized to higher order by

$$D_{\mathbf{x}_1,\mathbf{x}_2,...\mathbf{x}_n}K(\mathbf{v}) = \frac{\partial^n}{\partial\epsilon_1\partial\epsilon_2...\partial\epsilon_n}K(\mathbf{v} + \epsilon_1\mathbf{x}_1 + \epsilon_2\mathbf{x}_2 + ... + \epsilon_n\mathbf{x}_n)\Big|_{\epsilon_{1,2,...n}=0}. \tag{2.1.2}$$

We then say that a map is C^n or differentiable n times.

Given this concept of differentiability, there exists a special kind of map of interest called a **smooth** map, which must be continuous and infinitely differentiable (C^∞). We then call two sets M and N **diffeomorphic** if there exists a (C^∞) map $\phi : M \to N$ and a (C^∞) inverse $\phi^{-1} : N \to M$. The map ϕ is then called a **diffeomorphism**.

Next we define a **chart** or **coordinate system** as a subset U of M, along with a one-to-one map $\phi : U \to \mathbf{R}^n$ such that the image $\phi(U)$ is an open set in $\mathbf{R}^n$ and U is an open set in M. A C^∞ **atlas** is an indexed collection of charts $\{U_\alpha, \phi_\alpha\}$ which satisfies the two conditions:

1. The union of the U_α is equal to M, in other words the U_α cover M entirely.

2. The charts are smoothly sewn together, or in other words the intersections of any subsets $U_\alpha \cap U_\beta$ on the manifold map smoothly to $\mathbf{R}^n$ and all the associated maps are C^∞

This process of smoothly sewing together C^∞ maps to describe a manifold is illustrated in Figure 2.1.2. Finally, the C^∞ manifold is defined as a set M together

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Fig. 2.1.2.: A manifold can be described by overlapping coordinate charts.
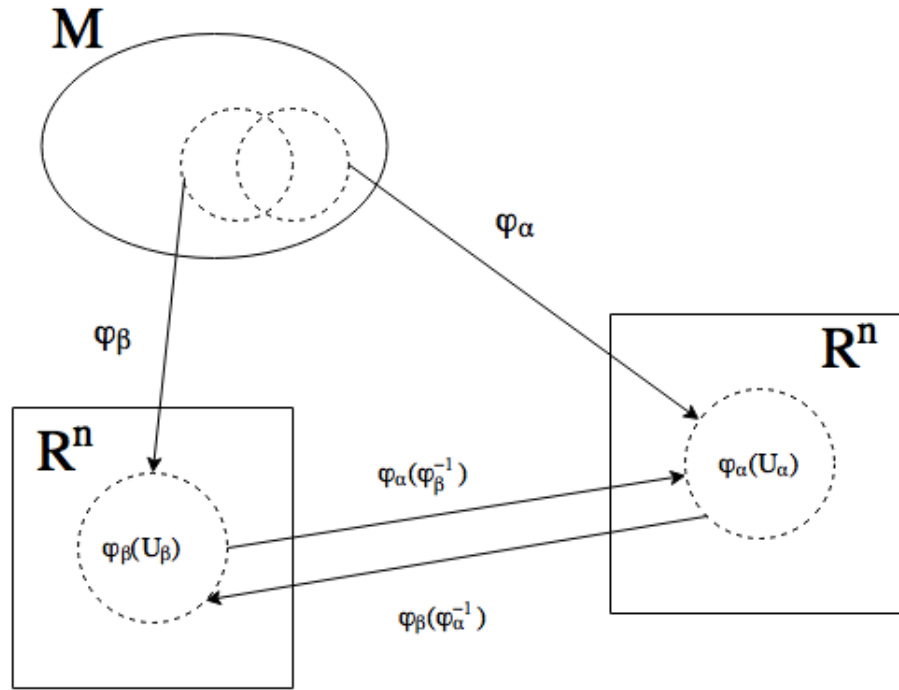
with an atlas that contains every possible chart. It is implicitly assumed hereinafter that all manifolds are smooth and differentiable at least as many times as we need them to be.

2.2 Vectors and Tangent Spaces

Now that we have a mathematical notion of "what is spacetime?", we now turn to the question of "what is in spacetime?" As in all of physics, we model the motion and interaction of objects using **vectors**. The concept of a vector is familiar, but we will introduce them in a way which generalizes to curved spaces effectively.

An important feature of the definitions we use is that they do not depend upon the manifold M being embedded in $\mathbf{R}^n$, in order to keep our results as general as possible. To this end, we define a **curve** α on M as a smooth map

$$\alpha : \mathbf{R} \to M \tag{2.2.1}$$

which assigns to each real number a unique point on the manifold. Then, let (U, ϕ) be a coordinate chart on M, with the assumption that $\alpha(t) \in U$ for all $t \in \mathbf{R}$. The curve α then has the **coordinate representation**

$$x^k = \phi^k(\alpha(t)) \tag{2.2.2}$$

where ϕ^k denotes the kth coordinate at the point $\phi(\alpha(t)) \in \mathbf{R}^n$.

For some fixed $t = t_0$ we can define the **tangent vector** at the point $\alpha(t_0)$ as the directional derivative of the real-valued function $f(\alpha(t))$ at $t = t_0$ using the chain rule, which gives

$$\frac{d}{dt} f(\alpha(t))\Big|_{t=t_0} = \frac{\partial f}{\partial x^k} \frac{dx^k}{dt}\Big|_{t=t_0}. \tag{2.2.3}$$

Now define the differential operator $\mathbf{v}$ as

$$\mathbf{v} = \sum_k v^k \frac{\partial}{\partial x^k}, \tag{2.2.4}$$

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where $v^k = \dfrac{dx^k}{dt} = \dfrac{d}{dt}\Big(\phi^k(\alpha(t))\Big)\Big|_{t=t_0}$. Then we can rewrite (2.2.3) as

$$\frac{d}{dt}f(\alpha(t))\Big|_{t=t_0} = v^k\frac{\partial f}{\partial x^k}, \tag{2.2.5}$$

where we have now introduced the **Einstein summation convention**. In this convention, it is implied that there is a sum over all possible values of an index which is repeated once up and once down. Additionally, an index can only be repeated exactly once up and once down. This convention will become more important when we introduce tensors in the next section.

From (2.2.5) we see that a vector can be thought of as not only a tangent to a curve, but also as a differential operator which acts on a smooth function f to give a number $\mathbf{v}(f)$. We then define the set of all possible tangent vectors at a point p in M as the **tangent space** of M at p, denoted $T_p(M)$. A useful way to conceptualize the tangent space is to visualize all of the possible curves on M that pass through the point p and then drawing their tangent vectors at that point. An example of this is shown in Figure 2.2.1.

It can be shown that the tangent space satisfies all properties of a vector space, but it is sufficient for our purposes to just take this fact for granted and move on. Given the fact that $T_p(M)$ is a vector space we rewrite (2.2.4) as

$$\mathbf{v} = v^k\mathbf{e}_k, \tag{2.2.6}$$

where $\mathbf{e}_k = \dfrac{\partial}{\partial x^k}$ are the **basis vectors** for the space.

An important consideration is that vectors are geometrical objects, and therefore must be independent of the coordinate system in which they are described. Thus we require that the components of vectors transform contravariantly, according to

$$v^{k'} = \frac{\partial x^{k'}}{\partial x^k}v^k. \tag{2.2.7}$$

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Fig. 2.2.1.: An example tangent space T_p for a two-sphere.
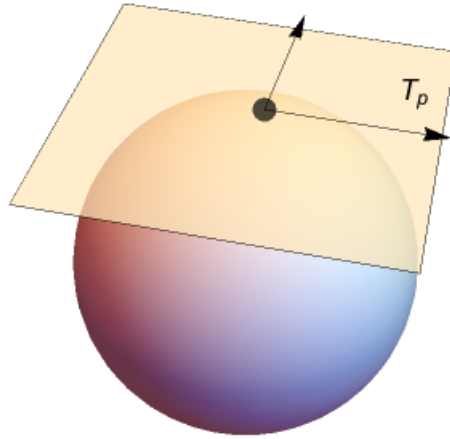
Since the tangent space is a vector space, it stands to reason that there exists a **dual vector space** (again the proof can be found in the reference texts), which is called the **cotangent space**, denoted $T_p^*(M)$. Each element of this space is a linear function ω which takes each vector $\mathbf{v}$ to a real number $\omega(\mathbf{v})$. These linear functions are called **dual vectors**, **covariant vectors**, **one-forms**, or even just "forms" for short.

Given a smooth function f, its differential df is in fact a one-form, which acts on vectors according to

$$df(\mathbf{v}) \equiv \langle df, v \rangle \equiv \mathbf{v}(f) = v^k \frac{\partial f}{\partial x^k} \tag{2.2.8}$$

where $\langle\,,\rangle$ denotes the inner product of a vector and a form. df has the coordinate representation

$$df = \frac{\partial f}{\partial x^k} dx^k. \tag{2.2.9}$$

Analogously to vectors, the set of dx^k form a **dual basis** for the cotangent space and we can write the components as $\omega^k = \dfrac{\partial f}{\partial x^k}$. Dual vectors can then be represented

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by

$$\omega = \omega_k dx^k \ . \tag{2.2.10}$$

Then the inner product of a vector and a form can be written as

$$\left\langle \omega_k dx^k, v^m \frac{\partial}{\partial x^m} \right\rangle = \omega_k v^m \left\langle dx^k, \frac{\partial}{\partial x^m} \right\rangle = \omega_k v^m \delta_m^k = \omega_k v^k. \tag{2.2.11}$$

Analogously to vectors, the transformation equation for the components of one-forms is given by

$$\omega_{k'} = \frac{\partial x^k}{\partial x^{k'}} \omega_k. \tag{2.2.12}$$

2.3 Tensors

To continue our discussion of objects in spacetime, we now turn to **tensors**. In short, tensors are multilinear maps defined at a point in the manifold, which generalize the concepts of vectors and forms. This section will define tensors in a rigorous way, utilizing both index-free and indexed (local coordinate) notation. Then, after discussing some of the properties of tensors and their components, we will discuss a very special tensor called the metric.

Begin with a differentiable manifold M of dimension n and consider a point p on the manifold. We call $\mathbf{V}$ a tensor of rank $\{1,0\}$ if $\mathbf{V}$ is a linear map assigning a unique real number to each one-form $\omega \in T_p^*(M)$. More specifically, linearity implies that

$$\mathbf{V}(\omega + \sigma) = \mathbf{V}(\omega) + \mathbf{V}(\sigma) \tag{2.3.1}$$

and

$$\mathbf{V}(c\omega) = c\mathbf{V}(\omega). \tag{2.3.2}$$

From this it should be clear that a tensor of rank $\{1,0\}$ is a contravariant vector.

Next, call $\mathbf{T}$ a tensor of rank$\{2,0\}$ at p, if $\mathbf{T}$ assigns a unique real number to

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each ordered pair of one-forms $\omega, \sigma \in T_p^*(M)$, and does so in a bilinear fashion. That is, that $\mathbf{T}$ is linear in each of its arguments separately:

$$\mathbf{T}(c_1\omega_1 + c_2\omega_2, \sigma) = c_1\mathbf{T}(\omega_1, \sigma) + c_2\mathbf{T}(\omega_2, \sigma). \tag{2.3.3}$$

and

$$\mathbf{T}(\omega, c_1\sigma_1 + c_2\sigma_2) = c_1\mathbf{T}(\omega, \sigma_1) + c_2\mathbf{T}(\omega, \sigma_2). \tag{2.3.4}$$

The collection of all such bilinear maps $\mathbf{T} : T_p^*(M) \times T_p^*(M) \rightarrow \mathbf{R}$ is called the **tensor product space** $\otimes^2 T_p(M)$. Analogously to the tangent space, it is the collection of all $\{2,0\}$ tensors at p. The tensor product space is also a vector space, with addition and scalar multiplication defined by

$$(\mathbf{T}_1 + \mathbf{T}_2)(\omega, \sigma) = \mathbf{T}_1(\omega, \sigma) + \mathbf{T}_2(\omega, \sigma) \tag{2.3.5}$$

and

$$(c\mathbf{T})(\omega, \sigma) = c(\mathbf{T}(\omega, \sigma)) \tag{2.3.6}$$

for all $c \in \mathbf{R}$.

We can then define a special kind of $\{2,0\}$ tensor $\mathbf{u} \otimes \mathbf{v} : T_p^*(M) \times T_p^*(M) \rightarrow \mathbf{R}$ as

$$(\mathbf{u} \otimes \mathbf{v})(\omega, \sigma) = \omega(\mathbf{u})\sigma(\mathbf{v}) \tag{2.3.7}$$

for all one-forms ω, σ. We call $\mathbf{u} \otimes \mathbf{v}$ the **tensor product** of $\mathbf{u}$ and $\mathbf{v}$, and of course this object is an element of the tensor product space.

To write this tensor as an indexed object, first consider the set of one-forms $\{\beta_k\}$ which constitutes a basis for $T_p^*(M)$. If we expand $\omega, \sigma \in T_p^*(M)$ in terms of this basis as $\omega = \omega_a\beta^a$ and $\sigma = \sigma_b\beta^b$ then, using the linearity property we can write

$$\mathbf{T}(\omega, \sigma) = \mathbf{T}(\omega_a\beta^a, \sigma_b\beta^b) = \mathbf{T}(\beta^a, \beta^b)\omega_a\sigma_b \equiv T^{ab}\omega_a\sigma_b \tag{2.3.8}$$

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where T^{ab} are the **components** of $\mathbf{T}$ relative to this basis. We will, often enough, refer to the components of a tensor and the tensor itself interchangeably, although splitting the tensor into components technically requires the extra structure of a set of basis objects. This is analogous to vectors, which are geometrical objects independent of a particular coordinate system, but whose components are not. Note that we will be using "index notation" and "abstract notation" interchangeably depending on the situation.

If we instead have a basis $\{\mathbf{b}_k\}$ for $T_p(M)$ it can similarly be shown that

$$\mathbf{T} = T^{ab}\mathbf{b}_a \otimes \mathbf{b}_b. \tag{2.3.9}$$

Thus, any tensor can be written as a linear combination of the tensor products of basis vectors.

To complete the definition of $\{2,0\}$ tensors we now want to determine how they transform. By using the definition of tensor components and the transformation property of differential forms we can write

$$T^{a'b'} = \mathbf{T}(dx^{a'}, dx^{b'}) = \mathbf{T}(\frac{\partial x^{a'}}{\partial x^a}dx^a, \frac{\partial x^{b'}}{\partial x^b}dx^b) = \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^{b'}}{\partial x^b}\mathbf{T}(dx^a, dx^b) \tag{2.3.10}$$

or equivalently

$$T^{a'b'} = \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^{b'}}{\partial x^b}T^{ab}. \tag{2.3.11}$$

This equation holds for any $\{2,0\}$ tensor. It is clear to see that this process of defining a $\{2,0\}$ tensor easily generalizes to tensors of type $\{n,0\}$. We will just have n upper indices and all of the properties are otherwise the same. Another name for an $\{n,0\}$ tensor is a **rank n contravariant tensor**.

Now we seek a definition of a $\{0,n\}$ tensor or a **rank n covariant tensor**. Begin with a $\{0,1\}$ tensor, which can be defined as a linear map which assigns a unique real

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number to each tangent vector $\mathbf{u} \in T_p(M)$. As we saw with {1,0} tensors (which were just vectors), {0,1} tensors are the same as forms. Next, define a {0,2} tensor as a bilinear map which assigns a unique real number $\mathbf{S}(\mathbf{u}, \mathbf{v})$ to each ordered pair of tangent vectors $\mathbf{u}, \mathbf{v} \in T_p(M)$ in a bilinear fashion, that is

$$\omega(\mathbf{u} + \mathbf{v}) = \omega(\mathbf{u}) + \omega(\mathbf{v}) \tag{2.3.12}$$

and

$$\omega(c\mathbf{u}) = c\omega(\mathbf{u}) \tag{2.3.13}$$

for any real number c.

We then call the collection of all such bilinear maps $\mathbf{S} : T_p(M) \times T_p(M) \to \mathbf{R}$ the tensor product space $\otimes^2 T_p^*(M)$. This is the collection of all rank 2 covariant tensors at a point p. Again, this tensor product space is a vector space with the usual definitions of addition and scalar multiplication. Similarly to the contravariant case, we can define the tensor product $\omega \otimes \sigma$, which is of course an element of the tensor product space.

If the set of tangent vectors $\{\mathbf{b}_k\}$ is a basis for $T_p(M)$, then we can expand a tensor in terms of components in this basis as

$$\mathbf{S}(\mathbf{u}, \mathbf{v}) = \mathbf{S}(u^a \mathbf{b}_a, v^b \mathbf{b}_b) = u^a v^b \mathbf{S}(\mathbf{b}_a, \mathbf{b}_b) = S_{ab} u^a v^b. \tag{2.3.14}$$

Alternatively, if the set of one-forms $\{\beta^k\}$ is a basis for $T_p^*(M)$ then we can expand to get

$$\mathbf{S} = S_{ab} \beta^a \beta^b \tag{2.3.15}$$

. So, any {0,2} tensor can be written as a linear combination of tensor products of basis one-forms. Next we write the transormation equation for a rank 2 covariant

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tensor as

$$S_{a'b'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} S_{ab}. \tag{2.3.16}$$

As was the case with contravariant tensors, the generalization to rank {0,n} is trivial.

Finally, we can write the definition of a **rank** $\{r, s\}$ **mixed tensor** as a multi-linear map

$$\mathbf{M} : (T_p^*(M))^r \times (T_p(M))^s \to \mathbf{R} \tag{2.3.17}$$

which maps each ordered $(r + s)$ tuple into a real number. This can be written as

$$\mathbf{M}(\omega_1, \omega_2, ...\omega_r, \mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_s) \in \mathbf{R}. \tag{2.3.18}$$

Similarly to the previous processes, it can be shown that $\mathbf{M}$ can be expanded in the form

$$\mathbf{M} = M_{k_1 k_2 ... k_r}^{j_1 j_2 ... j_r} \mathbf{b}_{j_1} \otimes \mathbf{b}_{j_2} \otimes ... \otimes \mathbf{b}_{j_r} \otimes \beta^{k_1} \otimes \beta^{k_2} \otimes ... \otimes \beta^{k_s}. \tag{2.3.19}$$

We refer to such a tensor as "rank r contravariant and rank s covariant" or even just "rank(r+s)". As was stated before, tensors are often referred to by only their indexed components for shorthand notation. The transformation equation follows the same pattern we have established, that is

$$M_{k_1' k_2' ... k_n'}^{j_1' j_2' ... j_n'} = \left(\frac{\partial x^{j_1'}}{\partial x^{j_1}} \frac{\partial x^{j_2'}}{\partial x^{j_2}} ... \frac{\partial x^{j_n'}}{\partial x^{j_n}}\right) \cdot \left(\frac{\partial x^{k_1}}{\partial x^{k_1'}} \frac{\partial x^{k_2}}{\partial x^{k_2'}} ... \frac{\partial x^{k_n}}{\partial x^{k_n'}}\right) M_{k_1 k_2 ... k_n}^{j_1 j_2 ... j_n}. \tag{2.3.20}$$

Next we turn to a discussion of the properties of tensors. Begin with the operation of **contraction**, which turns a $\{r, s\}$ tensor into a $\{r - 1, s - 1\}$ tensor. Contraction is just summing over one upper and one lower index, according to

$$S^{\mu\rho}{}_\sigma = T^{\mu\nu\rho}{}_{\sigma\nu}. \tag{2.3.21}$$

The result will always be a well defined tensor according to previous definitions. It is

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only permissible to contract and upper index with a lower index. Note that the order of the indices matters, so that in general

$$T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu}. \tag{2.3.22}$$

Tensors can also be **symmetric** in two indices (in this case μ and ν) if they satisfy

$$S_{\mu\nu\rho} = S_{\nu\mu\rho}. \tag{2.3.23}$$

Tensors can be symmetric in more than two indices. For the above example, if all possible combinations of μ, ν and ρ are equivalent, the tensor is symmetric in all three of those indices.

Similarly, a tensor is **antisymmetric** in any of its indices if it changes sign when those indices are exchanged. For example if

$$S_{\mu\nu\rho} = -S_{\mu\rho\nu} \tag{2.3.24}$$

then the above tensor is antisymmetric in ρ and ν. If a tensor is symmetric or antisymmetric in all of its indices we can refer to it as just "symmetric" or just "antisymmetric".

For a $\{1,1\}$ tensor we can define the **trace** as the scalar quantity:

$$X \equiv X^{\mu}{}_{\mu} = Tr(X). \tag{2.3.25}$$

To conclude the exposition of tensors, we now introduce what is probably the single most important object in general relativity: the **metric tensor**. The metric tensor is a very special symmetric $\{0,2\}$ tensor that has many different interpretations and functions. It extends the concept of a scalar or "dot product" in curved spacetime

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via

$$g(u, v) = u \cdot v. \tag{2.3.26}$$

Note that the above equation begins the convention of not writing vectors with arrows or bold face. As there are many different objects being manipulated in general relativity, it is common practice to "abuse notation" for the sake of simplicity. It is common in general relativity to see vectors written as either their components v^μ or just as the vector itself v. From the above, we see that the metric can give the length of a vector via

$$g(v, v) = v \cdot v = |v|^2. \tag{2.3.27}$$

Two vectors are orthogonal if

$$g(u, v) = u \cdot v = 0. \tag{2.3.28}$$

Given a basis e_i the dot product can also be written

$$g(u, v) = g(u^i e_i, v^j e_j) = g_{ij} u^i v^j \tag{2.3.29}$$

Another important function of the metric is to provide the notion of a **line element**, or infinitesimal distance in curved spacetime via

$$ds^2 = g_{ij} dx^i dx^j, \tag{2.3.30}$$

where dx^i and dx^j are basis forms. Quite often "line element" is used interchangeably with "metric", as the two are intimately related. In fact, given a line element

$$ds^2 = g_{11} dx^2 + 2g_{12} dx dy + g_{22} dy^2 \tag{2.3.31}$$

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we can read off the coefficients of the metric in matrix form as

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{2.3.32}$$

We say that an interval (or a vector) is **timelike** if $ds^2 < 0$, it is **spacelike** if $ds^2 > 0$, and it is **null** or **lightlike** if $ds^2 = 0$. We can define the locus of lightlike points emanating from a point p as the surface of a **light cone**. Then, any interval beginning at p and extending to a point inside this cone is timelike and any interval extending outside the cone is spacelike.

The metric has an inverse g^{ij} defined via

$$g^{ij}g_{jk} = g_{mk}g^{mi} = \delta^i_k. \tag{2.3.33}$$

The metric and its inverse also provide the important operation of **raising and lowering indices**. To raise an index use

$$T^{\alpha\beta\mu}{}_{\delta} = g^{\mu\gamma}T^{\alpha\beta}{}_{\gamma\delta} \tag{2.3.34}$$

and to lower an index use

$$T^{\alpha}{}_{\beta\mu\delta} = g_{\beta\gamma}T^{\alpha\gamma}{}_{\mu\delta}. \tag{2.3.35}$$

These roles that the metric fulfills are just a sampling of the most common operations. When we move on to curvature and the Einstein field equations, we will see that the metric even serves to replace the role of the Newtonian gravitational field ϕ.

2.4 Lie Derivatives

With vectors and tensors thoroughly defined, we move on to the methods of manipulating these objects. Since they are defined at particular points on manifolds, there is no obvious way to compare two vectors/tensors at two different points. Because a manifold can, in general, be curved arbitrarily the objects at the two points in question exist in different tangent spaces and thus cannot be logically compared to one another. In later sections this fact will help us to rationalize the idea of faster than light travel, but at the moment we would like a method of comparison. The **Lie derivative** fulfills this role, completely independent of the existence of the metric.

Consider a smooth vector vield v on a region of a manifold. The congruence of integral curves (with curve parameter λ) of this vector field defines a mapping ϕ of the manifold onto itself. We can identify a point p with another point q on the same integral curve using this mapping according to $q = \phi_\lambda(p)$. Consider another, independent, vector field u on the manifold with its own associated curve. If we take the curve associated with u and "drag" it from p to q using the congruence associated with v, as demonstrated in Figure 2.4.1 we have completed the act of **Lie dragging** the vector u.

The Lie derivative is then defined as the following: evlauate the vector at $q = \phi_\lambda(p)$, drag it back to p using the inverse map ϕ_λ^{-1}, and take the difference with the original vector at p in the limit $\lambda \to 0$. Mathematically this can be written as

$$\pounds_v u \equiv \lim_{\lambda \to 0} \left(\frac{\phi_\lambda^{-1}(u|_{\phi_\lambda(p)}) - u|_p}{\lambda} \right) \quad . \tag{2.4.1}$$

We can then write this component-wise as

$$\pounds_v u^\alpha = \frac{du^\alpha}{d\lambda} - u^\beta \partial_\beta v^\alpha \quad , \tag{2.4.2}$$

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Fig. 2.4.1.: Lie dragging. The vector field v is represented by the vertical curves, while the vector field u is represented by the horizontal curves
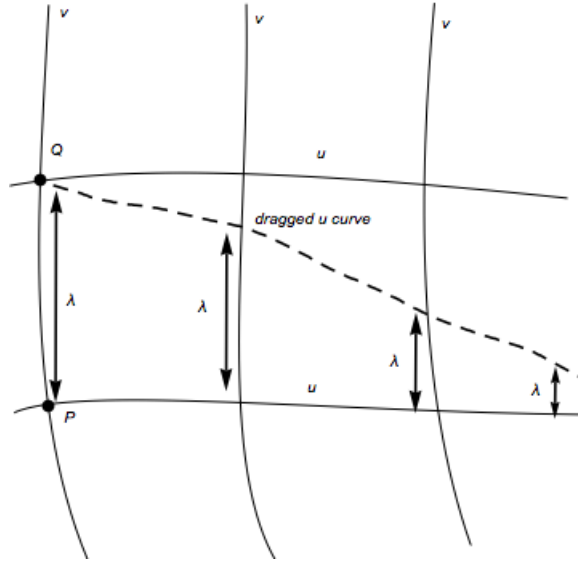
or equivalently

$$\pounds_v u^\alpha = v^\beta \partial_\beta u^\alpha - u^\beta \partial_\beta v^\alpha \quad . \tag{2.4.3}$$

The above can be recognized as the commutator of two vectors so that

$$\pounds_v u = [v, u] \quad . \tag{2.4.4}$$

The above definition applies to vectors and vector fields, and can easily be adapted to forms and form fields via

$$\pounds_v \omega_\alpha = v^\beta \partial_\beta \omega_\alpha + \omega_\beta \partial_\alpha v^\beta \quad . \tag{2.4.5}$$

From here, it is a simple matter to extend the Lie derivative to tensors and tensor fields of arbitrary rank by adding/subtracting a term for each index. As an example we have

$$\pounds_v T^\alpha{}_\beta = v^\mu \partial_\mu T^\alpha{}_\beta - T^\mu{}_\beta \partial_\mu v^\alpha + T^\alpha{}_\mu \partial_\beta v^\mu \quad . \tag{2.4.6}$$

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The Lie derivative can be thought of as a way to write partial derivatives along the direction of a given vector field in a way that is independent of the coordinates. Consider an **adapted coordinate system**, in which we adapt one of the coordinates of a given coordinate system, say x^1, to the curves of a vector field v. We will then have $x^1 = \lambda$ and $e_1 = v$, which implies that the components of v are given by $v^\alpha = \delta_1^\alpha$. Then the Lie derivative of a tensor $T^{\alpha\beta}$ will simplify to just

$$\pounds_v T^{\alpha\beta} = \partial_1 T^{\alpha\beta} \quad . \tag{2.4.7}$$

As an illustration of the importance of the Lie derivative in the context of symmetries of a given spacetime, consider a manifold with metric $g_{\alpha\beta}$. If the metric is invariant under Lie dragging with respect to some vector field ξ then we will have

$$\pounds_\xi g_{\alpha\beta} = 0 \quad . \tag{2.4.8}$$

If, for some metric $g_{\alpha\beta}$ there exists a vector field ξ that satisfies the above equation, we say that ξ is a **Killing vector field**. This concept is analogous to cyclic coordinates, which lead to conserved quantities in the Lagrangian formulation of classical mechanics. Killing vectors can be used to arrive at various useful equations involving symmetries of the spacetime in question, for example the orbits of celestial bodies in the Schwarzschild solution.

2.5 Curvature

Next on the quest to describe spacetime, we need to figure out how to characterize its curvature. The go to object for describing how things change is the partial derivative ∂_μ. The problem with the partial derivative on curved manifolds, however, is that it does not transform in a covariant manner (i.e. like a tensor should). We will therefore seek a covariant form of the derivative which will act on vectors and transform the way that a tensor operator should.

We begin by investigating the transformation properties of the partial derivative acting on a scalar function ϕ. Using our established transformation equation we arrive at

$$\partial_{\mu'}\phi = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \quad , \tag{2.5.1}$$

which transforms exactly as expected. The problem comes in when we allow the partial derivative to act instead on a vector v^ν. In this case we have

$$\partial_{\mu'}v^{\nu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu\right)\left(\frac{\partial x^{\nu'}}{\partial x^\nu}v^\nu\right). \tag{2.5.2}$$

Applying the product rule for the ∂_μ operator yields

$$\partial_{\mu'}v^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\nu \partial x^\mu}v^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\partial_\mu v^\nu. \tag{2.5.3}$$

The second term above looks like the transformation behavior of a tensor, but the first term ruins everything. We therefore define a **covariant derivative** to be an ordinary partial derivative plus a linear correction that guarantees the correct tensorial transformation properties, that is

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\lambda}v^\lambda \quad , \tag{2.5.4}$$

where the $\Gamma^\nu_{\mu\lambda}$ are called the **connection coefficients**, or the **Christoffel symbols**

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of the second kind. It is important to note that, by definition, the connection coefficients are not tensors. They are defined to transform such that there will be an extra (subtracted) term in order to cancel the unwanted term in (2.4.3) above. Thus, the covariant derivative of a vector transforms covariantly because we demanded it be so in defining it. Similarly we can define the covariant derivative of a one-form by demanding there be instead a subtracted linear correction so that

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \quad . \tag{2.5.5}$$

We can then define the covariant derivative of a tensor with any number of indices. For each upper index introduce one $+\Gamma$ term and for each lower index introduce one $-\Gamma$ term. It's useful to add that there are different conventions for denoting partial and covariant derivatives, one of which uses a semicolon directly before an index to indicate a covariant derivative and a comma before an index to indicate a partial derivative. Examples would be

$$v^\mu_{,\nu} = \partial_\nu v^\mu \quad . \tag{2.5.6}$$

$$T^\mu_{\alpha;\beta} = \nabla_\beta T^\mu_\alpha \quad . \tag{2.5.7}$$

For this discussion, the "∇" notation will usually be used, but in some cases the semicolon notation may be used as well.

As might be expected, since the metric tells us about the geometry of a manifold, the connection coefficients can be found for a given metric. However, to get the desired form useful for general relativity, we must make two assumptions. The first assumption requires another definition. Note that while the connection itself is not a tensor, the difference between two connections is a tensor. Define the **torsion tensor** to be

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad . \tag{2.5.8}$$

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The torsion tensor above is clearly antisymmetric in its lower indices, and therefore we say that any connection symmetric in its lower indices is "**torsion-free**", since the torsion tensor will vanish for that connection. Next we define a connection to be **metric compatible** if the covariant derivative of the metric (and its inverse) with respect to that connection is zero, i.e.

$$\nabla_\rho g_{\mu\nu} = 0 \quad . \tag{2.5.9}$$

In general relativity, we typically use the **metric compatible torsion-free connection**, which satisfies the above two properties. An important feature of this connection is that it is unique: there is exactly one per metric. These requirements are not part of the definition of the covariant derivative, they merely identify a unique one from many possible other definitions.

Finally, with this information in hand, we can find an expression for the metric compatible torsion-free connection in terms of derivatives of the metric itself:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \quad . \tag{2.5.10}$$

We can achieve an intuitive understanding of the connection coefficients by thinking of them as gravitational "forces" in the context of general relativity. Since it is always possible to pick a coordinate system at a specific point on a manifold for which the connections vanish (just by requiring that the first derivatives of the metric vanish at that point), one could then rationalize that this coordinate system corresponds to the reference frame of a body in free fall. This concept of making the connection coefficients vanish at a point only applies at that point, and not in its neighborhood, which will come into play when we next discuss geodesics.

In order to understand curvature and what a covariant derivative implies, we need to discuss and define **geodesics**. Simiply put, a geodesic is a "straight line" on

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a curved manifold. An example would be great circles on the surface of a two-sphere. If we were to take one of those great circles off of the sphere, cut it in the middle and lay it out on a table, it would be perfectly straight. For a more mathematically rigorous definition, start by considering a parameterized curve $x^\mu(\lambda)$ and a tensor $K^{\alpha\beta}{}_\nu$. In order for this tensor to be constant along the curve in flat space we simply require (via the chain rule)

$$\frac{d}{d\lambda}K^{\alpha\beta}{}_\nu = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}K^{\alpha\beta}{}_\nu = 0 \quad . \tag{2.5.11}$$

To generalize this property to curved spaces we simply replace the partial derivative operator with a covariant derivative. Define the **directional covariant derivative** to be

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu \quad . \tag{2.5.12}$$

Then the **parallel transport** of the tensor along the path $x^\mu(\lambda)$ is just the requirement that the covariant derivative of the tensor along the path vanishes, i.e.

$$\frac{D}{d\lambda}K^{\alpha\beta}{}_\nu \equiv \frac{dx^\sigma}{d\lambda}\nabla_\sigma K^{\alpha\beta}{}_\nu = 0 \quad . \tag{2.5.13}$$

The above is the parallel transport equation and tells us about how vectors transport in curved spaces. To arrive at the geodesic equation, we simply demand that the directional covariant derivative of a tangent vector along the path $x^\mu(\lambda)$ vanish so that

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 \quad , \tag{2.5.14}$$

or equivalently by applying our earlier definition

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \quad . \tag{2.5.15}$$

The above is the **geodesic equation**, whose solutions are geodesics on the man-

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ifold. Note that in flat space the connection coefficients vanish and the geodesic equation reduces to the equation for a straight line. In general relativity, we say that a **test particle** is a particle which does not influence the geometry though which it moves (a great approximation for small objects orbiting a large one for example). Test particles always move along geodesics, or the path of greatest proper time along a manifold.

Next on the path to describing the curvature of a manifold, we finally come to the fundamental object describing curavature: the **Riemann curvature tensor**. First, consider the commutator of two covariant derivatives $[\nabla_\mu \nabla_\nu]v^\rho$. This commutator tells us the difference between parallel transporting v^ρ first in one direction and then in another. Expanding the commutator we can write

$$[\nabla_\mu \nabla_\nu]v^\rho = \nabla_\mu\nabla_\nu v^\rho - \nabla_\nu\nabla_\mu v^\rho \quad . \tag{2.5.16}$$

Using the definition of the covariant derivative it can be shown that the above expands to

$$[\nabla_\mu \nabla_\nu]v^\rho = (\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})v^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda v^\rho \quad . \tag{2.5.17}$$

Recognizing that the last term above has the torsion tensor $2\Gamma^\lambda_{[\mu\nu]}$ and assuming zero torsion, we can write the Riemann curvature tensor as

$$[\nabla_\mu \nabla_\nu]v^\rho = R^\rho{}_{\sigma\mu\nu}v^\sigma \quad . \tag{2.5.18}$$

Then in explicit form, in terms of the (torsion-free) connection, the Riemann curvature is

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad . \tag{2.5.19}$$

An important fact to note about the Riemann tensor is that if it vanishes, we can

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always construct a coordinate system in which the metric components are constant, and vice-versa. This means that if each component of the Riemann tensor vanishes, the spacetime in question is flat.

At first glance, since there are four indexes and (in 4-dimensional spacetimes) four possible values for each index, it appears we have $4^4 = 256$ independent components of the curvature tensor. In fact, there are some symmetries of the curvature tensor which reduce that number from 256 to just 20 at maximum. These symmetries are more easily expressed using the form of the curvature tensor with all indices lowered; and are given by

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} \quad , \tag{2.5.20}$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad , \tag{2.5.21}$$

and

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \quad , \tag{2.5.22}$$

where in the last equation we have taken cyclic permutations of the last three indices. Another important identity involving the Riemann curvature tensor is the **Bianchi identity**, which is

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad , \tag{2.5.23}$$

where the square brackets around λ, ρ, and σ indicate a sum over cyclic permutations of those indices. Without the "bracket notation", the Bianchi identity is

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0 \quad . \tag{2.5.24}$$

With the Riemann curvature defined and some identities laid out, we can then define two other important curvature-related objects. First we define the **Ricci ten-**

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sor as the contraction of the Riemann tensor

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} \quad , \tag{2.5.25}$$

and next we define the **Ricci scalar** or **scalar curvature** as the trace of the Ricci tensor

$$R \equiv R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \quad . \tag{2.5.26}$$

The Ricci tensor is of course symmetric, due to the symmetries of the Riemann tensor.

The last curvature-related tensor to be discussed for this section comes from contracting the Bianchi identity twice with the inverse metric to write

$$0 = g^{\nu\sigma} g^{\mu\lambda} (\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu}) \quad . \tag{2.5.27}$$

Writing in terms of the Ricci tensor and Ricci scalar yields

$$0 = \nabla^{\mu} R_{\rho\mu} + \nabla_{\rho} R + \nabla^{\nu} R_{\rho\nu} \quad , \tag{2.5.28}$$

or equivalently

$$\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R \quad . \tag{2.5.29}$$

We can use the metric to rewrite the above as

$$\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} R g_{\mu\rho} \nabla^{\mu} \quad . \tag{2.5.30}$$

Factoring out the covariant derivative and renaming indices ($\rho \to \nu$) we can write

$$\nabla^{\mu} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0 \quad . \tag{2.5.31}$$

Then we define the **Einstein tensor** or the **trace-reversed Ricci tensor** to be the part in parenthesis above

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad . \tag{2.5.32}$$

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The Einstein tensor is also symmetric due to the symmetry of the Ricci tensor. For later use, we can rewrite the twice-contracted Bianchi identity in the form

$$\nabla^\mu G_{\mu\nu} = 0 \quad . \tag{2.5.33}$$

2.6   The Einstein Field Equations

With a means to describe the curvature of spacetime, we next need an under-
standing of how this curvature arises as a result of the presence of matter and energy
density. The equations that provide this description are called the **Einstein Field
equations**, and are the fundamental equations governing general relativity. They can
be thought of as the curved-spacetime analog of Poisson's equation for the Newtonian
potential.

Before describing the physical motivation for arriving at the particular form of
the field equations, we will first need a tensorial description of the matter-energy
density at a point in spacetime. The object which provides this description is the
stress-energy tensor. This tensor represents the four-momentum density seen by
any observer, and can be represented in matrix form as

$$T^{\mu\nu} = \begin{pmatrix} \rho & j^1 & j^2 & j^3 \\ j^1 & S^{11} & S^{12} & S^{13} \\ j^2 & S^{21} & S^{22} & S^{23} \\ j^3 & S^{31} & S^{32} & S^{33} \end{pmatrix} . \tag{2.6.1}$$

In the above, ρ represents relativistic mass density, the j^i represent flux of relativistic
mass across the x^i surface, and the S^{ij} represent the flux of the i th component of
linear momentum across the x^j surface.

A particular form of matter that is often used as a heuristic tool is the **perfect
fluid**, which is comprised of many point-particles that are affected by gravitation,
but do not interact with one another. This perfect fluid has a very tidy stress energy
tensor associated with it, which is often used in simplified calculations to elucidate

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the properties of a spacetime. The stress energy tensor for a perfect fluid is given by

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \quad , \tag{2.6.2}$$

where p represents pressure of the fluid.

The stress energy tensor is always symmetric in general relativity, and obeys a conservation equation wherein the covariant divergence of the stress energy tensor vanishes

$$\nabla_\nu T^{\mu\nu} = 0 \quad . \tag{2.6.3}$$

With a working definition of the matter-energy density through a region of spacetime in tow, we next demonstrate a method for postulating the Einstein field equations. Since the stress energy tensor is rank 2, it is reasonable to guess that wee seek a tensor equation involving two-index objects with the stress energy tensor on one side, and some combination of curvature tensors on the other. This equation must reduce to Poisson's equation for the Newtonian potential

$$\nabla^2 \phi = 4\pi G \rho \quad , \tag{2.6.4}$$

in the non relativistic limit. One such tensor equation is

$$R_{\mu\nu} = \kappa T_{\mu\nu} \quad . \tag{2.6.5}$$

While this equation does reduce to Poisson's equation in the non-relativistic limit, there is a problem in the fact that the right hand side must obey the conservation equation (2.6.3). This means that the left hand side has additional constraints given

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by

$$\nabla_\nu R^{\mu\nu} = 0 \quad , \tag{2.6.6}$$

which creates an overdetermined and unsolvable system. There is another rank two curvature-related tensor, however, which involves both the Ricci curvature tensor and the metric tensor: the Einstein tensor. From (2.4.33) we already know that the Einstein tensor obeys the requisite conservation equation that the stress energy tensor must also obey, and thus no additional constraints are imposed on the system. It is therefore reasonable to propose a field equation of the form

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad . \tag{2.6.7}$$

Choosing κ appropriately will allow the above to collapse into Poisson's equation in the non-relativistic flat space limit. A little calculation shows that we must take (in $c = 1$ or "geometrical" units) $\kappa = 8\pi G$ and the Einstein field equations can then be written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \quad , \tag{2.6.8}$$

where the G on the right hand side of the above is the Newtonian gravitational constant (not the trace of the Einstein tensor).

We can take the trace of the above equation to find that $R = -8\pi G T$ Substituting this result and moving the trace term to the right hand side of the equation we can recast the field equations as

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \quad . \tag{2.6.9}$$

In vacuum, where $T_{\mu\nu} = 0$, the right hand side of the above vanishes and we can write the **vacuum Einstein field equations** as

$$R_{\mu\nu} = 0 \quad . \tag{2.6.10}$$

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2.7   The 3+1 split

As should be expected with a set of relativistic equations, the Einstein field equations treat space and time on equal footing. While this is mathematically natural, it is sometimes more physically understandable to separate space and time. Among other things, this allows one to treat time as an informal "line" along which three dimensional cross sections or "snapshots" of space can be "slid". These foliations of an n dimensional manifold into a collection of $(n-1)$ dimensional submanifolds are called **hypersurfaces**.

We can then identify three possible types of hypersurfaces in general relativity. A **spacelike hypersurface** is a hypersurface whose normal vector points in a timelike direction, a **timelike hypersurface** is a hypersurface whose normal vector points in a spacelike direction, and lastly a **null hypersurface** is a hypersurface whose normal vector points in the null direction. This definition for a null hypersurface can be a bit misleading, though, since a normal vector to a null surface is also tangent to that surface. This is because $k \cdot k = 0$ for a null vector and so the concepts of normal and tangent become slightly obscured for the null case. What we really want is for a normal vector k to satisfy $k \cdot \ell = 0$ for any vector ℓ on the surface.

This splitting of spacetime into surfaces of constant time has been done in a few different ways with various goals in mind. The 3+1 formalism happens to be the most common, and is often used in setting up and solving initial value problems in numerical relativity for the purpose of studying the dynamics of certain systems. While this thesis is not directly concerned with numerical relativity, the formalism of the 3+1 split will help later to clearly illustrate the behavior of warp drives, along with another formalism (discussed in the final chapter) which separates spacetime into light cones. An additional clarification of notation should be made at this point: As

Fig. 2.7.1.: The 3+1 split formalism. The line to $(x^i)'$ represents the normal line of the Eulerian observers, while the line to x^i represents the constant coordinate line.
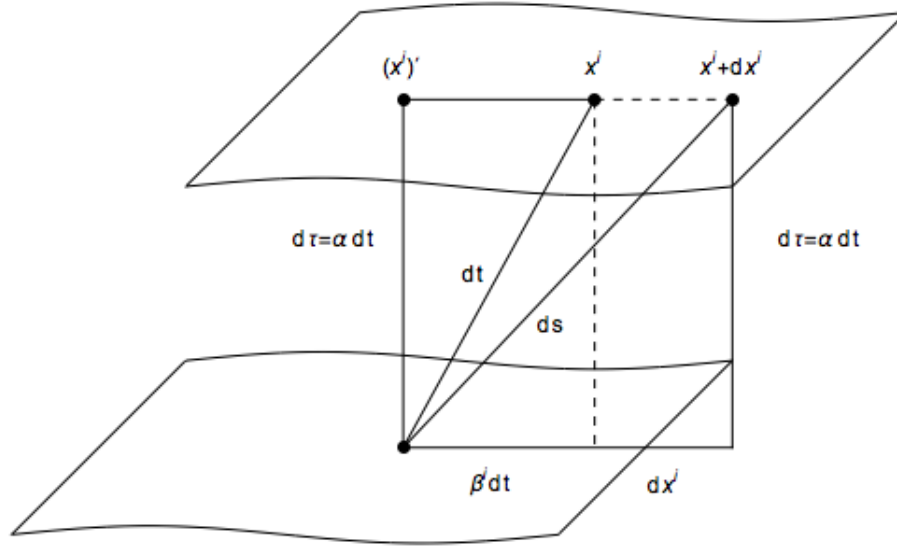
in many standard texts on relativity, we will assume Latin indices are purely spatial (1,2,3), while Greek indices include both space and time components (0,1,2,3).

Begin by considering a spacetime with metric $g_{\mu\nu}$ and two adjacent spacelike hypersurfaces Σ_t and Σ_{t+dt} From Figure 2.7.1, which demonstrates the foliation of spacetime into spacelike hypersurfaces, we can determine the geometry of spacetime between the two hypersurfaces with three pieces of information:

1. The three dimensional metric γ_{ij} measuring distances within the hypersurface itself

$$d\sigma^2 = \gamma_{ij}dx^i dx^j \quad . \tag{2.7.1}$$

2. The lapse of proper time $d\tau$ between the hypersurfaces as measured by an observer moving along the direction normal to the hypersurfaces

$$d\tau = \alpha(t, x^i)dt \quad , \tag{2.7.2}$$

where α is called the **lapse function** and such observers are called **Eulerian**

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observers.

3. The relative velocity β^i between the Eulerian observers and the lines of constant spatial coordinates

$$x^i_{t+dt} = x^i_t + \beta^i(t, x^j)dt \quad , \tag{2.7.3}$$

where the 3-vector β^i is called the **shift vector**.

With these three pieces of information specified, and using the above figure we can write the 3+1 split metric in general form

$$ds^2 = (\alpha dt)^2 + \gamma_{ij}(\beta^i + dx^i)(\beta^j + dx^j) \quad , \tag{2.7.4}$$

or equivalently

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j \quad . \tag{2.7.5}$$

Additionally from the figure we can write the four velocity of the Eulerian observers as the timelike unit vector normal to the hypersurfaces

$$n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right) \quad , \quad n_\mu = (-\alpha, 0, 0, 0) \quad . \tag{2.7.6}$$

2.8 Projection Tensors and Extrinsic Curvature

Up to now, we have mostly considered the curvature of Riemannian manifolds without mention of any kind of embedding in a higher dimensional space. In differential geometry we often avoid embeddings in order to keep the mathematics as general as possible. With the 3+1 split, however, knowledge of embeddings or "slices" of spacetime is required.

When a manifold is embedded in a higher dimensional space there are actually two kinds of curvature that can be distinguished: **intrinsic curvature** and **extrinsic curvature**. Intrinsic curvature is the curvature that has been discussed up to this point, and does not depend on any particular kind of embedding. This curvature is measurable by "inhabitants" of the surface, and is also detectable by inhabitants of a higher dimensional space; take for example the curvature of a two-sphere. Extrinsic curvature, on the other hand, is only detectable by inhabitants of the higher dimensional embedding space. An example of an object with purely extrinsic curvature (no intrinsic curvature) would be a cylinder.

As we have seen, the intrinsic curvature is defined in terms of the metric tensor. The extrinsic curvature, on the other hand, is defined as a measure of the change of a normal vector n^μ to the hypersurface under parallel transport. For a mathematical treatment of the extrinsic curvature, we first define the **projection operator** or **projection tensor**, which projects arbitrary vectors onto the hypersurfaces, as

$$P^\mu_\nu \equiv \delta^\mu_\nu + n^\mu n_\nu \quad . \tag{2.8.1}$$

The projection tensor can also be written in fully covariant form as

$$P_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad . \tag{2.8.2}$$

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The projection tensor has a few interesting properties aside from the ability to project vectors onto hypersurfaces. First, given two vectors V^μ and W^ν that are already tangent to the hypersurface, the projection tensor will actually act just like the metric tensor, via

$$P_{\mu\nu}V^\mu W^\nu = g_{\mu\nu}V^\mu W^\nu + n_\mu n_\nu V^\mu W^\nu \quad . \tag{2.8.3}$$

The second term above vanishes due to orthogonality and we simply have

$$P_{\mu\nu}V^\mu W^\nu = g_{\mu\nu}V^\mu W^\nu \quad . \tag{2.8.4}$$

Additionaly it can be shown from the original definition that the projection tensor is **indempotent**, which means that acting two or more times produces the same result as only acting once, i.e.

$$P^\mu_\alpha P^\alpha_\nu = P^\mu_\nu \tag{2.8.5}$$

The projection tensor is also known as the **first fundamental form** of the hypersurface. Additionally, due to the first property discussed, the projection tensor is none other than the spatial metric γ_{ij} from the 3+1 split formalism (for spacelike hypersurfaces).

Defining the extrinsic curvature as a measure of the change of a normal vector n^μ under parallel transport, it becomes clear that we can characterize this curvature mathematically with a covariant derivative. It tells us how the orientation of the surface changes from place to place, via

$$K_{\mu\nu} \equiv -P^\alpha_\mu \nabla_\alpha n_\nu = -(\nabla_\mu n_\nu + n_\mu n^\alpha \nabla_\alpha n_\nu) \quad . \tag{2.8.6}$$

The extrinsic curvature is purely spatial as defined above, and also happens to be

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symmetric. The extrinsic curvature can also be defined in terms of the Lie derivative and the spatial metric γ_{ij} from the 3+1 split via the equation

$$K_{\mu\nu} = -\frac{1}{2}\pounds_n \gamma_{\mu\nu} \quad . \tag{2.8.7}$$

Since the Lie derivative is linear such that

$$\pounds_{\phi n}\gamma_{\mu\nu} = \phi\pounds_n\gamma_{\mu\nu} \tag{2.8.8}$$

we can write

$$K_{\mu\nu} = -\frac{1}{2\alpha}\pounds_{\alpha n}\gamma_{\mu\nu} = -\frac{1}{2\alpha}(\pounds_t - \pounds_\beta)\gamma_{\mu\nu} \quad , \tag{2.8.9}$$

where t is a timelike vector and β is the shift vector from the 3+1 split metric. We can then make use of an adapted coordinate system $\pounds_t \to \partial_t$ to write

$$\partial_t \gamma_{ij} - \pounds_\beta \gamma_{ij} = -2\alpha K_{ij} \quad . \tag{2.8.10}$$

Expanding using the definition of the Lie derivative we have

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \quad , \tag{2.8.11}$$

where D_i represents covariant differentiation only with respect to the spatial metric γ_{ij}. In fact, this derivative can be defined in terms of the projection of the the full covariant derivative as $D_\mu \equiv P^\alpha_{\ \mu}\nabla_\alpha$. The above can then be rearranged to read

$$K_{ij} = \frac{1}{2\alpha}(D_i\beta_j + D_j\beta_i - \partial_t g_{ij}) \quad . \tag{2.8.12}$$

2.9 Energy Conditions

Considering the right hand side of the Einstein field equations, one sees that there is no built-in restriction on the type of matter/energy described by the stress energy tensor $T_{\mu\nu}$. It is clear that the metric serves the role of solution to the field equations, but there is no way to specify which metrics lead to realistic descriptions of spacetime. Indeed, one could simply pick an arbitrary metric, compute the relevant curvature components and then demand that the Einstein tensor be equal to the stress energy tensor. The spacetime described may or may not make any practical sense, however. Thus, we can see the motivation for the energy conditions, which are coordinate invariant restrictions on the stress energy tensor, and define what a "realistic" source of energy and momentum should be.

The energy conditions are not physical laws, but instead they are a set of assuptions about how any reasonable form of matter/energy density should behave. As such, there are many formulations of the energy conditions. Most commonly they are presented in the form:

1. **Weak Energy Condition** (WEC): The energy density seen by all observers is non-negative, so that

$$T_{\mu\nu}u^{\mu}u^{\nu} \geq 0 \tag{2.9.1}$$

    for any timelike unit vector u^{μ} .

2. **Strong Energy Condition** (SEC): The energy density plus the sum of the principal pressures must be non-negative, so that

$$T_{\mu\nu}u^{\mu}u^{\nu} + T/2 \geq 0 \quad , \tag{2.9.2}$$

    for an any timelike unit vector u^{μ}, with $T \equiv T^{\mu}{}_{\mu}$. For a perfect fluid this

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becomes $\rho + 3p \geq 0$. The Einstein field equations imply that this statement is equivalent to

$$R_{\mu\nu}u^{\mu}u^{\nu} \geq 0 \quad . \tag{2.9.3}$$

3. **Null Energy Condition** (NEC): The energy density plus any of the principal pressures must be non-negative, so that

$$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0 \quad , \tag{2.9.4}$$

for any null vector k^{μ} . For a perfect fluid this becomes $\rho + p \geq 0$, which implies that the energy density can be negative so long as there is a compensating positive pressure. Through the field equations it can be shown that this statement is equivalent to

$$R_{\mu\nu}k^{\mu}k^{\nu} \geq 0. \tag{2.9.5}$$

There are two additional common energy conditions, the **Dominant Energy Condition** (DEC) and the **Null Dominant Energy Condition** (NDEC). The DEC includes the WEC as well as the additional requirement that $T_{\mu\nu}u^{\mu}$ be a non-spacelike vector, so that

$$T_{\mu\nu}T^{\nu}{}_{\lambda}u^{\mu}u^{\lambda} \geq 0 \quad . \tag{2.9.6}$$

This implies for our perfect fluid model that $\rho \geq |p|$ so that the energy density must be non-negative and greater than the magnitude of the pressure.

The NDEC is the DEC for null vectors. It states that for any null vector k^{μ} that $T_{\mu\nu}u^{\mu}$ must be a non-spacelike vector, so that

$$T_{\mu\nu}T^{\nu}{}_{\lambda}k^{\mu}k^{\lambda} \geq 0 \quad . \tag{2.9.7}$$

CHAPTER 3

WARP DRIVE SPACETIMES

With the relevant mathematical frameworks reviewed, we move to a discussion of the past and current literature regarding warp drives. We begin with a summary of the orignal paper by Alcubierre introducing the warp drive metric[11], and a method of rewriting it[12], then discuss an example trip in a spaceship using this metric as a guideline[13], and finally move to a discussion of the literature that spawned surrounding it.

There are a number of physical problems regarding the nature of warp drives, which can be described using a few different viewpoints of relativistic physics. This chapter will essentially be a review of these problems as seen through these different "lenses". Note that other theoretical methods of superluminal travel, such as traversable wormholes and Krasnikov tubes [14, 15], exist and even have some similarities to warp drives, but are not discussed in detail.

3.1 The Alcubierre Warp Drive

An important idea to reiterate and expand upon at this point is that the speed of light is only the local speed limit in general relativity. As we have seen, there is no natural way to compare two vectors or tensors at different points on a manifold, since they exist in separate tangent spaces. We cannot compare the velocity at point A to the velocity at point B unless points A and B happen to be sufficiently close together that the curvature in their vicinity vanishes. This is the limiting case where general relativity collapses to special relativity, and is a great approximation

for measurements in laboratories. Across larger distances, within the framework of general relativity, the cosmic speed limit has no fundamental definition, and so warp drive spacetimes can be feasibly considered.

Ordinarily, when one seeks a solution of the Einstein field equations, one would assume a reasonable distribution of matter (probably following the conventions of the energy conditions) and then work out the resulting curvature induced on spacetime. The Alcubierre metric is an example of doing just the opposite: we assume a particular form of the curvature to achieve the desired effect (superluminal travel), only to find that the matter-energy distribution giving rise to this spacetime does not obey the energy conditions. While the fact that warp drives and related spacetimes violate the energy conditions[11, 16–19] classically limits the possibility of actually building them, we can nonetheless use them as a tool for probing the theoretical foundations and limits of general relativity. We will see in later sections that the possibility of negative energy densities is not forbidden in the framework of quantum field theory; and so the warp drive spawned considerable interest in exploring quantum restrictions in this context[20–25].

The first mention of a warp drive spacetime in the context of general relativity was by Miguel Alcubierre in 1994 [11]. The basis for the idea is that, since the cosmic speed limit need only be obeyed locally in general relativity, in principle one could move from point A to point B (separated by a proper distance D) in a coordinate time $T < \dfrac{D}{c}$, given an appropriate spacetime structure. Alcubierre's initial insight was to realize that one could contract spacetime in front of a spaceship and expand spacetime behind it in order to simultaneously push the ship away from point A and pull the ship toward point B. Alcubierre achieved this effect by starting with the

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general 3+1 split metric

$$ds^2 = -d\tau^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j \quad . \tag{3.1.1}$$

Assuming the spaceship moves purely along the x axis of a cartesian coordinate system one can then use the definitions

$$\alpha = 1 \tag{3.1.2}$$

$$\beta^x = -v_s(t)f(r_s(t)) \tag{3.1.3}$$

$$\beta^y = \beta^z = 0 \tag{3.1.4}$$

$$\gamma ij = \delta_{ij} \quad , \tag{3.1.5}$$

where

$$v_s(t) = \frac{dx_s(t)}{dt} \qquad r_s(t) = [(x - x_s(t))^2 + y^2 + z^2]^{1/2} \quad , \tag{3.1.6}$$

and where

$$f(r_s(t)) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\,\tanh(\sigma R)} \tag{3.1.7}$$

is called the shape function; with $x_s(t)$ an arbitrary fucntion describing the trajectory along which the spaceship is pushed, and with $R > 0$ and $\sigma > 0$ arbitary parameters. Using the above definitions, the metric can be written in the simple form

$$ds^2 = -dt^2 + (dx - v_s(t)f(r_s)dt)^2 + dy^2 + dz^2 \quad . \tag{3.1.8}$$

From the definition of the 3-metric $\gamma_{ij} = \delta_{ij}$ we see that the 3-geometry of the hypersurfaces is flat. Since the lapse is simply $\alpha = 1$, it can be seen that the "Eulerian observers" (those whose 4-velocity is normal to the hypersurfaces) are in free-fall, i.e. that timelike curves normal to the hypersurfaces are geodesics. Since the shift vector is non-uniform, the spacetime is not overall flat; however it can be shown that the shift vector vanishes for $r_s \gg R$ and thus that far away from the

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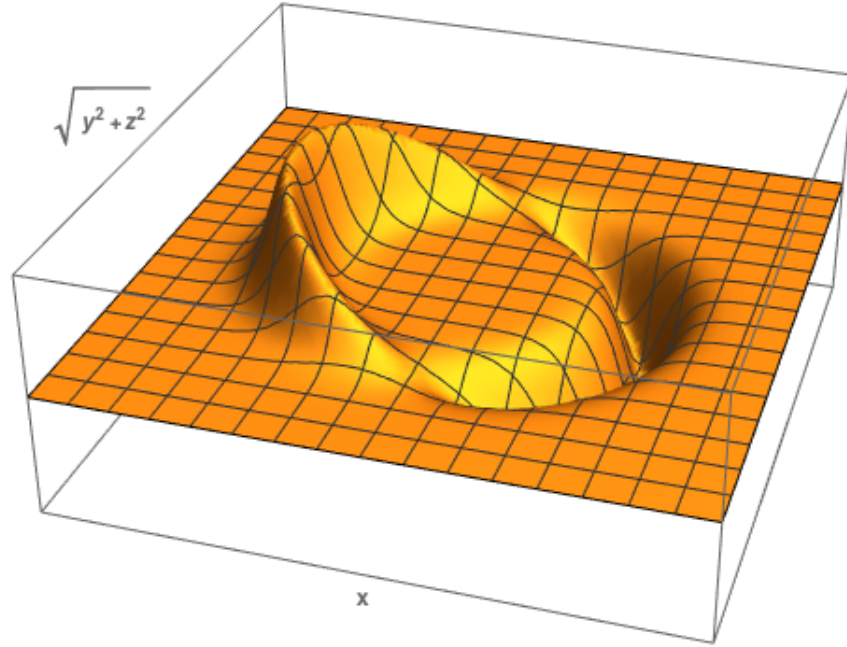
region centered at $(x_s(t), 0, 0)$ the spacetime will be essentially flat. This region with radius $\sim R$ corresponds to the radius of the "warp bubble", inside of which would be the spaceship. The parameter σ can be thought of as being inversely proportional to the thickness of the warp bubble's walls.

Since the 3-geometry of the hypersurfaces is flat, the curvature of the spacetime is purely extrinsic. From the definition of the spatial metric γ_{ij} the expression for the extrinsic curvature reduces to just

$$K_{ij} = \frac{1}{2}(\partial_i \beta_j + \partial_j \beta_i) \quad . \tag{3.1.9}$$

Next, to describe the expansion θ of the volume elements of the Eulerian observers we take the trace of the extrinsic curvature according to

$$\theta = -\alpha K^i_{\ i} = v_s \frac{x_s}{r_s} \frac{df}{dr_s} \quad . \tag{3.1.10}$$

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The expansion has been visualized using software in Figure 3.1.1, taking the ship to be moving in the x direction and the parameters to be $\sigma = 3$ and $R = 2.2$.

It is also mentioned in Alcubierre's paper that making the substitution $x_s(t) = x$ into the metric provides the result $d\tau = dt$, which implies that the spaceship moves along a timelike curve, and furthermore that the spaceship experiences no time dilation as it moves.

Another, less encouraging fact about the Alcubierre metric is that it violates the weak, strong, and dominant energy conditions. Using the fact that the (timelike) 4-velocity of the Eulerian observers is given by

$$u^\mu = \frac{1}{\alpha}(1, -\beta^i) \qquad u_\mu = -(\alpha, 0) \quad , \tag{3.1.11}$$

one can then calculate the Einstein tensor and show that the energy density seen by these observers would be

$$T^{\mu\nu} u_\mu u_\nu = \alpha^2 T^{00} = \frac{1}{8\pi} G^{00} = -\frac{1}{8\pi} \frac{v_s^2(x^2 + y^2)}{4r_s^2} \left(\frac{df}{dr_s}\right)^2 \quad . \tag{3.1.12}$$

The above expression is always negative, which shows that the weak and dominant energy conditions are violated. In a similar fashion, it can be shown that this spacetime also violates the strong energy condition. Calculations of the Einstein tensor components and the subsequent violations of the energy conditions are detailed in the appendices of a 2004 paper by Lobo and Visser[26], and in the review text by Lobo[13]. The energy density has been visualized in Figure 3.1.2

Fig. 3.1.2.: The matter/energy density of the alcubierre metric. Note that both regions are negative, violating the weak energy condition
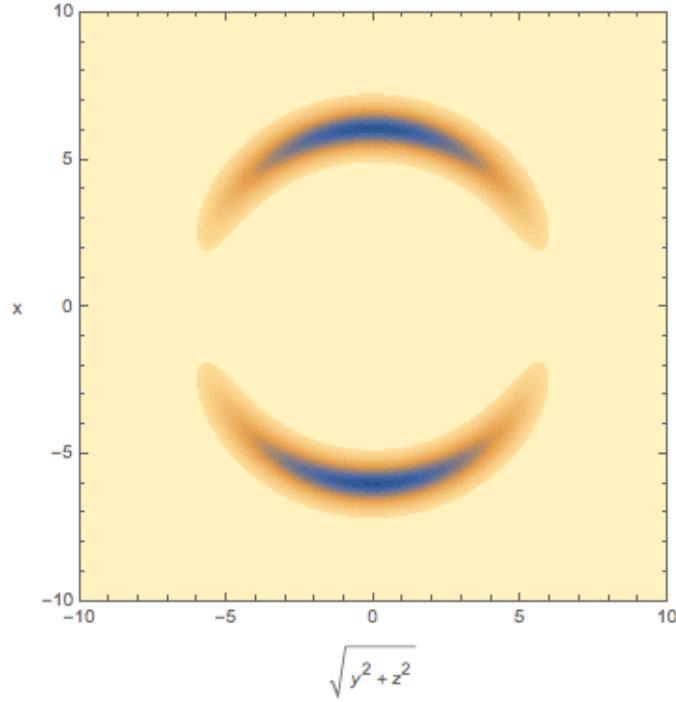
3.2 The Zero-Expansion Warp Drive

Earlier it was stated that the assumed mechanism by which the warp drive operates is the contraction/expansion of the space ahead/behind the spaceship, and this expansion was visualised in Figure 3.1.1. While this expansion of volume elements may be a feature of the Alcubierre metric, José Natário[12] has demonstrated that it is not necessarily a fundamental feature of warp drive spacetimes. To briefly illustrate this concept, begin by considering a more general warp drive metric of the form

$$ds^2 = -dt^2 + \sum_{i=1}^{3}(dx^i - X^i dt)^2 \quad , \tag{3.2.1}$$

where the vector field $\mathbf{X}$ is given by

$$\mathbf{X} = X^i \frac{\partial}{\partial x^i} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} \quad . \tag{3.2.2}$$

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The above vector field can be thought of as simply a time dependent vector in Euclidean space. Thus, the warp drive metric above is uniquely specified by the choice of this vector field. We can think of this metric as just the Alcubierre metric, but with the ship capable of moving in an arbitrary direction. The curvature of this spacetime will still be purely extrinsic, and we can recover the Alcubierre metric by choosing appropriate components for the vector $\mathbf{X}$, namely

$$X = v_s(t)f(r_s) \qquad , Y = Z = 0 \quad , \tag{3.2.3}$$

with $v_s(t)$ and r_s defined the same as the Alcubierre metric, and $f(r_s)$ any smooth function approximating a step function in a neighborhood of the origin and equal to zero in the neighborhood of infinity.

Next note that the expansion of the volume element associated with the Eulerian observers is given by

$$\theta = K^i_{\ i} = \partial_i X^i = \nabla \cdot \mathbf{X} \quad . \tag{3.2.4}$$

This means that if $\mathbf{X}$ has vanishing divergence, then the expansion/contraction of spacetime also vanishes. To demonstrate an example of this, assume the warp bubble moves with velocity $v_s(t)\dfrac{\partial}{\partial x}$ and adopt spherical coordinates with

$$e_r \equiv \frac{\partial}{\partial r} \qquad , e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \qquad , e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \quad . \tag{3.2.5}$$

Choosing $\mathbf{X} = X^r e_r + X^\theta e_\theta + X^\phi e_\phi$ and $f(r) = \dfrac{1}{2}$ for large r and $f(r) = 0$ for small r it can be shown that the relevant components of the extrinsic curvature become

$$K_{rr} = -2v_s f' \cos\theta \quad , \tag{3.2.6}$$

$$K_{\theta\theta} = v_s f' \cos\theta \quad , \tag{3.2.7}$$

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and

$$K_{\phi\phi} = v_s f' \cos\theta \quad . \tag{3.2.8}$$

The expansion is then the sum of these components

$$\theta = K_{rr} + K_{\theta\theta} + K_{\phi\phi} = 0 \quad , \tag{3.2.9}$$

and so we see that the expansion vanishes due the contraction in the radial direction being compensated by expansions in the angular directions. Note, however that although the expansion has been made to vanish, when one works out the energy density as seen by Eulerian observers, the expression is still everywhere negative and thus the energy conditions are still violated for this expansionless warp drive spacetime. These calculations are laid out in detail in the appendices of a 2004 paper by Lobo and Visser[26].

3.3 A Journey Through Space

With the basics of the warp drive laid out, let us take a moment to ignore the physical problems with this concept and enjoy an example trip from our solar system at point A to some distant star system at point B, as described in [11, 13].

Consider two stars A and B separated by a distance D in a flat spacetime. At time t_0 the spaceship moves away from A at speed $v < 1$ (of course we are still using the usual relativistic convention that $c = 1$) using its conventional rocket engines. When the ship is a safe distance d away from the Earth, it can then come to rest and engage its warp engine (using its ample supply of "Unobtanium") to set up an appropriate disturbance in spacetime according the the Alcubierre metric. We will assume that $R \ll d \ll D$ where R is the radius of the warp bubble. Since the spaceship starts from rest, $v_s = 0$ and the disturbance will develop smoothly from the otherwise flat spacetime.

When the ship is halfway between the two stars, the disturbance would need to be modified such that the coordinate acceleration (recall that the ship experiences no proper acceleration) switches quickly from a to $-a$. The spaceship would then find itself at rest a distance d away from star B, and could use its conventional rocket engines to move through flat spacetime to its destination.

If the changes in acceleration are more or less instantaneous, then the total coordinate time T elapsed in the one way trip is just

$$T = 2\left(\frac{d}{v} + \sqrt{\frac{D - 2d}{a}}\right) \quad . \tag{3.3.1}$$

The above coordinate time will also be the proper time elapsed from the viewpoint of the stars A and B, since they remain in flat space. The proper time measured

on the spaceship, however, is given by

$$T = 2 \left(\frac{d}{\gamma v} + \sqrt{\frac{D - 2d}{a}} \right) \quad , \tag{3.3.2}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2}}$. This demonstrates that the only time dilation in this situation comes from the initial and final parts of the trip, where the ship is not using the warp engine. Maintaining the assumption that $R \ll d \ll D$ we can write the proper time as

$$\tau \approx T \approx 2\sqrt{\frac{D}{a}} \quad . \tag{3.3.3}$$

It is then clear to see that T can be made arbitrarily small by making a arbitrarily large. The proper time from both reference frames is then arbitrarily small and the spaceship can effectively travel faster than light. As has been shown before, the spaceship will always be on a timelike trajectory and inside its local light cone. That is, light itself will be pushed along by the distortion of spacetime. Note that moving a distance d before the warp engine is engaged is not necessary, but does ensure that the stars remain unaffected by the disturbance in spacetime, and therefore do not experience any related time dilation. This scenario of course assumes that we have some method of generating the warp bubble and ignores many other physical problems with the metric, some of which will be discussed in the next sections.

3.4 Classical Energy Estimations and Quantum Inequalities

As we saw in previous sections, the Alcubierre and Natário drives appear to require negative energy density in order to achieve the desired curvature of spacetime, something that is classically forbidden. In the regime of quantum field theory (QFT) however, violations of the classical energy conditions are abound, and have even been experimentally verified in the case of the Casimir effect.

The Casimir effect is a purely quantum mechanical phenomenon wherein conducting objects (the usual example being parallel conducting plates) at close distances experience a force between them, due to the change in the zero point energy of the electromagnetic field between the objects compared to the large vacuum surrounding them. What this translates to for our purposes is that small regions of negative energy density are indeed possible, and in theory could produce something resembling the curvature of a warp drive. The question to ask, then, is how much negative mass can we produce and for how long? We will first answer this question from a classical viewpoint, and then move on to a quantum inequality that gives upper bounds on the negative mass based on an energy-time uncertainty argument.

To first find a classical approximation of how much energy condition violating mass is needed to produce a warp bubble of a given size, thickness, and velocity, begin with the expression for the matter density of the warp bubble in terms of the stress energy tensor

$$\rho_{\text{warp}} = T_{\mu\nu}u^{\mu}u^{\nu} \quad , \tag{3.4.1}$$

where u^{μ} is the four-velocity of the Eulerian observers. To get the total mass in the bubble integrate this expression to get

$$M_{\text{warp}} = \int T_{\mu\nu}u^{\mu}u^{\nu} \quad d^3x \quad . \tag{3.4.2}$$

Recall that the energy density (in units of $G = c = 1$) is given by

$$\rho = T^{\mu\nu} u_\mu u_\nu = T^{00} = \frac{1}{8\pi} G^{00} = -\frac{1}{8\pi} \frac{v_b^2(x^2 + y^2)}{4r_s^2} \left(\frac{df}{dr_s} \right)^2 \quad , \tag{3.4.3}$$

where f is the Alcubierre shape function and v_b represents the (approximately constant) velocity of the warp bubble. Substituting this into the integral we have

$$M_{\mathrm{warp}} = -\frac{v_b^2}{32\pi} \int \frac{x^2 + y^2}{r_s^2} \left(\frac{df}{dr_s} \right)^2 r^2 \, dr \, d^2\Omega \quad , \tag{3.4.4}$$

or equivalently

$$M_{\mathrm{warp}} = -\frac{v_b^2}{12} \int \left(\frac{df}{dr} \right)^2 r^2 dr \quad . \tag{3.4.5}$$

The above integral can be approximated[26] using the Alcubierre shape function to get the result

$$M_{\mathrm{warp}} \approx -v_b^2 R^2 \sigma \quad . \tag{3.4.6}$$

Since σ is inversely proportional to the thickness of the warp bubble, we see that as the thickness decreases, the negative mass becomes greater. Additionally, exceeding the speed of light becomes more and more difficult based on this estimate.

Moving to the realm of QFT, an estimate of the lower bound on the negative energy density of a warp bubble was found by Pfenning and Ford[25] using results from work by Ford and Roman[27]. This work placed a limit on the magnitude and duration of the negative energy density experienced by an observer, in the form of a "quantum energy inequality" (QEI), given by

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu} u^\mu u^\nu \rangle}{\tau^2 + \tau_0^2} d\tau \geq -\frac{3}{32\pi^2 \tau_0^4} \quad , \tag{3.4.7}$$

where τ is an inertial observer's proper time and τ_0 is an arbitrary sampling time. This places a limit on the magnitude and duration of the negative energy density seen by an observer. This inequality was originally derived for a massless scalar field in

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Minkowski spacetime, but it can be argued[24] that it applies for curved spacetime if the sampling time is of or below the order of the smallest local radius of curvature. First define the thickness Δ of the warp bubble by rewriting the original Alcubierre shape function into the form

$$
f_{p.c.}(r) = \begin{cases} 1 & R < R - \dfrac{\Delta}{2} \\[2mm] -\dfrac{1}{\Delta}(r - R - \dfrac{\Delta}{2}) & R - \dfrac{\Delta}{2} < r < R + \dfrac{\Delta}{2} \\[2mm] 0 & r > R + \dfrac{\Delta}{2} \end{cases} \tag{3.4.8}
$$

We can find a relationship between Δ and σ by setting the two functions equal to one another at $r = R$, so that

$$
\Delta = \frac{\left[1 + \tanh^2(\sigma R)\right]^2}{2\sigma \tanh(\sigma R)} \quad . \tag{3.4.9}
$$

In the limit of large σR we have $\Delta \sim \dfrac{2}{\sigma}$. Substituting the expression for the energy density and integrating as done by Pfenning and Ford[25] provides the interesting intermediate result

$$
\Delta \leq 10^2 v_b L_{\text{Planck}} \quad , \tag{3.4.10}
$$

where Δ is the thickness of the warp bubble, and L_{Planck} is the Planck length.

Thus it can be seen that the wall thickness cannot be much above the Planck scale unless the velocity of the warp bubble is extremely large. Pfenning and Ford then go on to assume a bubble radius of 100 meters and find that the total negative energy required is approximately

$$
E \leq -3 \times 10^{20} \, M_{\text{galaxy}} v_b \quad , \tag{3.4.11}
$$

where M_{galaxy} is the mass of a typical galaxy. This means that even traveling at just the speed of light $v_b = 1$ the warp bubble appears to require negative energy ten

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orders of magnitude greater than the total mass of the entire visible universe.

This astounding number was reduced greatly in a paper by Van de Broeck[28] by rewriting the metric and effectively splitting the original Alcubierre drive into a few regions, some of which have positive energy and some of which have negative energy. It is demonstrated that the negative energy required for this spacetime (for $v_b = 1$) is roughly

$$E \leq -1.4 \times 10^{30} \text{ kg} \quad , \tag{3.4.12}$$

which is of the order of magnitude of a solar mass. This result is interesting, but brings up a bigger and more important question: If we can simply rewrite the existing metric and change the energies involved by such massive amounts, how can we be certain that any calculated numbers are reliable? This illustrates the fundamental motivation for attempting to consider warp drive spacetimes, or indeed any other spacetimes, in a coordinate-independent manner. In the next few sections we continue to briefly review other attempts to "break" the warp drive spacetime, some of which are actually coordinate-independent.

3.5 Linearized Warp Drives

Linearized gravity[7] is an approximation method for weak-field gravitation that has been used to describe gravitational radiation where the source is far away from the field being considered. The first step is to rewrite the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \tag{3.5.1}$$

where $\eta_{\mu\nu}$ represents the Minkowski metric and $h_{\mu\nu} << 1$ is a small perturbation on the background flat space. Next adopt the **Hilbert-Lorentz gauge** (also known by many other names)

$$\partial_\nu \bar{h}^{\nu\mu} = 0 \quad , \tag{3.5.2}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{3.5.3}$$

is the trace-reverse of the perturbation $h_{\mu\nu}$. Then we can write the linearized Einstein equations as

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu} \quad , \tag{3.5.4}$$

where $\Box = \partial_\mu \partial^\mu = h_{\alpha\beta}\partial^\alpha \partial^\beta h_{\mu\nu}$. We can equivalently write this in terms of the trace-reversed stress-energy tensor as

$$\Box h_{\mu\nu} = -16\pi\, G\bar{T}_{\mu\nu} \quad , \tag{3.5.5}$$

where

$$\bar{T}_{\mu\nu}(\vec{y}, \tilde{t}) = T_{\mu\nu}(\vec{y}, \tilde{t}) - \frac{1}{2}\eta_{\mu\nu}T(\vec{y}, \tilde{t}) \quad . \tag{3.5.6}$$

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This has a formal solution in terms of Greens functions given by

$$h_{\mu\nu}(\vec{x}, \tilde{t}) = 16\pi G \int d^3 y \frac{\bar{T}_{\mu\nu}(\vec{y}, \tilde{t})}{|\vec{x} - \vec{y}|} \quad , \tag{3.5.7}$$

where we have defined $\vec{x}$ to be the observation point, $\vec{y}$ to be the source point, and $\tilde{t} = t - |\vec{x} - \vec{y}|$ to be the retarded time, which accounts for travel time from the source to the observation point.

The implicit assumptions in this solution are that there is no incoming gravitational radiation, and that the global (not just local) geometry of the spacetime is approximately flat. It is this approximate global flatness that will allow us to break our earlier rule and compare vectors at completely different points in spacetime, to a good approximation. Following the argument made by Visser et. al.[16, 17], we consider a null vector k^μ of the unperturbed Minkowski spacetime so that

$$\eta_{\mu\nu} k^\mu k^\nu = 0 \quad . \tag{3.5.8}$$

This vector has a norm satisfying

$$|k|^2 = g_{\mu\nu} k^\mu k^\nu \quad , \tag{3.5.9}$$

or equivalently in our linearized theory

$$|k|^2 = \eta_{\mu\nu} k^\mu k^\nu + h_{\mu\nu} k^\mu k^\nu = h_{\mu\nu} k^\mu k^\nu \quad . \tag{3.5.10}$$

Using our general solution for $h_{\mu\nu}$ yields

$$h_{\mu\nu} k^\mu k^\nu = 16\pi G \int d^3 y \frac{\bar{T}_{\mu\nu}(\vec{y}, \tilde{t}) k^\mu k^\nu}{|\vec{x} - \vec{y}|} \quad . \tag{3.5.11}$$

Assuming the null energy condition holds, and noting that $|\vec{x} - \vec{y}|^{-1}$ is positive definite, we can see that the integrand itself is positive-definite and therefore the integral must be positive. Thus, ignoring the case of a completely empty spacetime

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we will have

$$g_{\mu\nu}k^{\mu}k^{\nu} > 0 \quad , \tag{3.5.12}$$

which shows that a null vector in the background flat metric must be spacelike in the full perturbed metric. This means that the null cone of the perturbed metric must everywhere fit inside of the null cone of the flat metric, i.e. the light cones always contract if the NEC holds. This demonstrates that the local coordinate speed of light can only decrease, and thus shows in a coordinate-independent manner that a warp drive could only exist via exotic matter *in the weak-field regime.*

We could then argue that this should likely hold in the strong field case as well, since we would expect that "starting the warp engine" should vary smoothly from flat space to the strong field perturbation. The catch to this argument, however, is that we can no longer assume global flatness in the strong field case and must use other methods to make our comparisons of the light cones, which considerably complicates the situation.

Using an identical process to our classical estimation for the mass of the warp bubble, it can be shown[13, 26] that for the linearized warp drive, where the bulk of the mass in question will just be the mass of the ship M_{ship} the classical inequality is

$$v^2 R^2 \sigma \leq M_{\text{ship}} \quad . \tag{3.5.13}$$

As one might expect for a weak gravitational effect, any reasonably sized ship will have its velocity severely limited.

3.6   Horizon Problems

We now turn to yet another lens through which to examine warp drive spacetimes. We first examine the light cone structure and show that the pilot of the spaceship in an Alcubierre metric cannot create or otherwise control in any way the warp bubble; and then demonstrate that an appropriate coordinate transformation results in a singularity and event horizon.

Consider the proper reference frame of an observer at the center of a warp bubble. Assume the warp bubble to be moving in the z direction and use a coordinate transformation $z' = z - z_0(t)$ so that the metric can be simplified to[13]

$$ds^2 = -dt^2 + dx^2 + dy^2 \left[dz' + (1 - f)vdt\right]^2 \quad , \tag{3.6.1}$$

where $f = f(r)$ is the original Alcubierre shape function. A photon emitted along the $+z$ direction will have $ds^2 = dx = dy = 0$ so that the above can be rearranged to get

$$\frac{dz'}{dt} = 1 - (1 - f)v \tag{3.6.2}$$

If the spaceship is at rest at the center of the bubble, then a photon in the "old" coordinates will have

$$\frac{dz}{dt} = v + 1 \tag{3.6.3}$$

and in the transformed coordinates (where $f = 1$ in the center)

$$\frac{dz'}{dt} = 1 \quad . \tag{3.6.4}$$

Now consider what happens when the photon moves to the region where spacetime is warped i.e. the "edge" of the warp bubble, at some point $z' = z'_c$. Its coordinate velocity can, in principle, be any value due to the curvature and so if we

Fig. 3.6.1.: A photon can approach, but never reach the edge of the warp bubble.
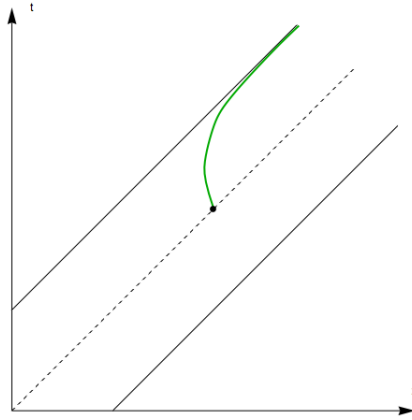
try setting it equal to zero we have

$$0 = 1 - (1 - f)v \quad . \tag{3.6.5}$$

So that when

$$f = 1 - \frac{1}{v} \tag{3.6.6}$$

the photon will be at rest relative to the warp bubble and is trapped within, effectively defining a horizon at the edges of the warp bubble.

If photons emitted in the interior region will never reach the outer region, then the outer region must lie outside the future light cone of the spaceship, which renders the Alcubierre metric quite unfeasible as a method of travel. In the crudest of terms: you need a warp drive to control your warp drive. Figure 3.6.1 shows a spacetime diagram of the situation, demonstrating the worldlines of an observer at rest that encounters the warp bubble; and also the worldline of a photon that is emitted from the center.

We can illustrate the event horizons of the warp drive metric in a different way by again considering a warp bubble moving only in the z direction for which $dx = dy = 0$

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and rewriting the metric in the two dimensional form

$$ds^2 = -(1 - v^2 f^2)dt^2 - 2vf\, dzdt + dz^2 \quad . \tag{3.6.7}$$

This time we will use our roughly constant bubble velocity v_b along with a location $r^2 = (z - v_b t)^2$. We can then use a coordinate transformation $r = z - v_b t$ or equivalently $dz = dr + v_b dt$ to write the line element as

$$ds^2 = -A(r)\left[dt - \frac{v_b(1 - f)}{A(r)}dr\right]^2 + \frac{dr^2}{A(r)} \quad , \tag{3.6.8}$$

where $A(r)$ is the Hiscock function defined as

$$A(r) = 1 - v_b^2(1 - f)^2 \quad . \tag{3.6.9}$$

We can next diagonalize the metric by using the coordinate transformation

$$d\tau = dt - \frac{v_b(1 - f)}{A(r)}dr \quad . \tag{3.6.10}$$

Then the metric takes the simpler form

$$ds^2 = -A(r)d\tau^2 + \frac{dr^2}{A(r)} \quad . \tag{3.6.11}$$

By noting that $A(r) \to 1$ for $r \to 0$ we see that τ is just the proper time of the observer at the center of the warp bubble. Next consider some point r_0 toward the outer edge of the warp bubble (analagous to our earlier point z'_c) that satisfies

$$f(r_0) = 1 - \frac{1}{v_b} \quad \text{and} \quad A(r_0) = 0 \quad . \tag{3.6.12}$$

At this point, there is a coordinate singularity and event horizon, provided that $v_b > 1$.

CHAPTER 4

## GENERALIZED WARP DRIVE SPACETIMES

Up to this point, there has been much discussion of the implications and unphysicality of the Alcubierre and related metrics, including descriptions of the nature of matter giving rise to these metrics. Most of the literature regarding warp drives and superluminal travel postulates a particular form of the metric and the resulting spacetime is then characterized by "back checking", looking at the stress energy tensor and doing related calculations. In principle, we could instead try to define how faster than light travel should work starting from some simple diagrams and discussion, and characterize the spacetime based on that. The focus of this chapter is on presenting the foundations for some potentially new ways of thinking about superluminal travel.

First, an alternative means of superluminal travel is described, somewhat analogous to the "Krasnikov Tube"[14], which sets the stage for the "2+2" splitting of spacetime. This 2+2 split could, in principle, be used to characterize the matter-energy involved in superluminal travel without the explicit need for a metric. This formalism utilizes a coordinate system based on the foliation of spacetime into light cones, which propagate along a timelike worldline. This method has some similarities with the Newman Penrose formalisim[29].

4.1 The "Warp Tube"

First assume that before the superluminal device is "turned on", that our spacetime is approximately Minkowskian, and can be described through the usual coordinates x, y, z, t . Take the Earth to be located at point A with spatial coordinates

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Fig. 4.1.1.: Spacetime diagram illustrating the boundaries of $D^+(S)$
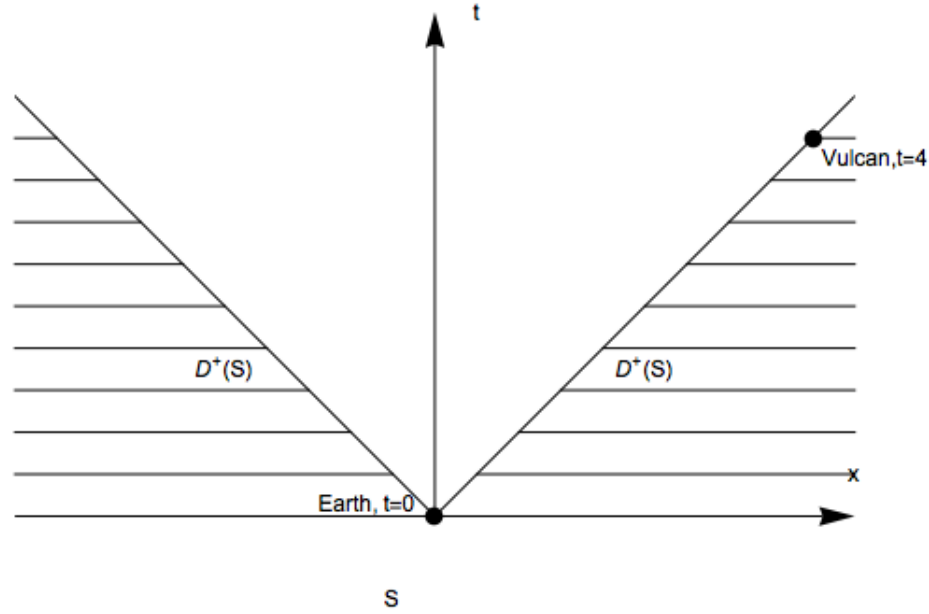
$x = y = z = 0$ and the planet Vulcan be located 4 light years away at point B. Suppose that at time $t = 0$ we turn on the warp drive, powered by some form of exotic matter.

On a spacelike hypersurface S at some $t < 0$ the spacetime is approximately flat and unaffected by the warp drive, and as such has evolution equations that obey the local light speed barrier. The future evolution of S is called the future domain of dependence of S, or $D^+(S)$. Because these evolution equations have no way to take our warp drive into account, they cannot extend into the future light cone of point A. The boundary of the future light cone of point A is therefore the boundary of $D^+(S)$ and is determined by the initial data on the surface S. The warp drive has no way of affecting $D^+(S)$ and cannot change its boundary. It takes four years for that boundary to reach point B and so we cannot travel there in less than four years no matter what the warp drive does within the boundary. This is illustrated in Figure 4.1.1. The instant "on-demand" warp drive cannot exist without tachyonic matter

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Fig. 4.1.2.: Spacetime diagram illustrating the action of the "warp tube"
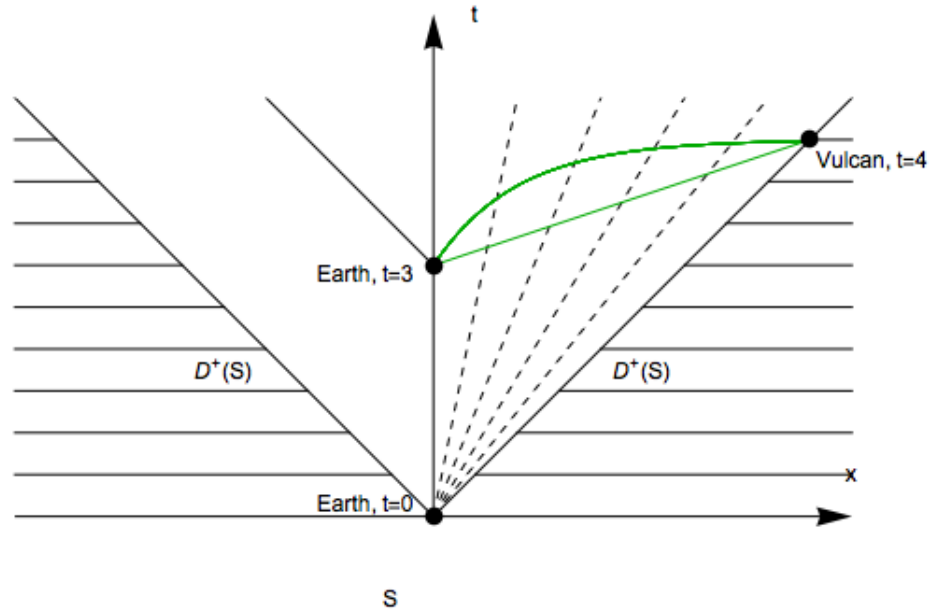
emmiters.

There is a different possibility for defining faster than light travel, however. We could instead build a "warp highway" or "warp tube" which, once built, would allow us to travel to a specific location in a very short time. This idea was first proposed by Krasnikov[14] by essentially writing a different version of the Alcubierre metric (which also happens to violate the energy conditions). In this next example, the future light cone of the "turn on" event (Earth at point A) still will not be affected, but an event at a later time, say $t = 3$ will have its light cone widened by some exotic matter.

Again take Earth to be at $x = y = z = 0$ and Vulcan to be four light years away. This time, however, at time $t = 0$ we begin to emit some finite amount of exotic matter, beginning the process of building our superluminal highway. One important fact to note is that our exotic matter is not travelling faster than light, and therefore we must wait for it to permeate the space between point A and point B. For the sake of simplicity, assume that the exotic matter can travel at the speed of light. At

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some later time $t = 3$, when the exotic matter has had time to (almost) reach the destination, we can send our spaceship along the warp tube. If the "warp tube" has been traveling at the speed of light, then we can reach the other planet at the absolute minimum time $t = 4$, with a total trip time of $\Delta t = 1$. Then, since the highway has been constructed, we can make trips back and forth between the Earth and the other planet as we please.

We can characterize this behavior using the Einstein clock synchronization method. Consider a two dimensional surface that includes the world lines of the origin, the destination, and the spaceship. We can locate events by a pair of null coordinates u, v. The times and distances that Einstein clock synchronization would assign to these events would be

$$t' = a(u + v), \qquad x' = a(u - v) \quad , \tag{4.1.1}$$

where a is some arbitrary constant that could be adjusted for clock rates. If we were to assume equal travel times, say $t_1 = t_2$ then the only way for this prescription for superluminal travel to work is by shrinking the distance to the destination. This implies moving solar systems, however, which seems a bit unreasonable.

We could instead think of the coordinate t' as being misaligned with global time so that the coordinate description of the outgoing and incoming signals through the warp tube will be asymmetrical. This effectively means that the warp tube would provide extremely fast travel one way, and extremely slow travel the other way. A rudimentary analogy would be to imagine trying to go the wrong way on a moving walkway in the airport.

One consideration about this description is that if we adjust this diagram for the frame of a moving observer, they could see up to three copies of the spaceship at the same time. They would first see the ship waiting at the Earth before departure,

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Fig. 4.1.3.: Spacetime diagram of the warp tube in a moving reference frame. The spaceship's trip in the tube has been approximated as a straight. The dashed lines correspond to constant time lines for the moving observer.
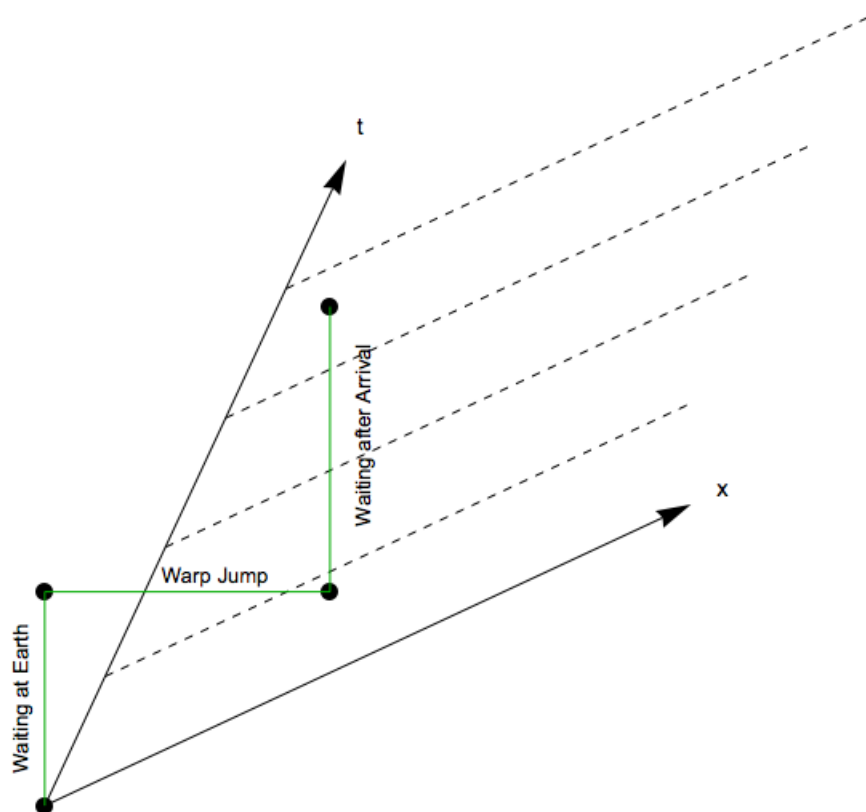
then see the event in which the ship arrives (at the same time that it is still waiting). Then at a later time the observer could simultaneously see the ship waiting before departure, making the warp tube jump backwards in time, and see the ship waiting after arrival. This consquence of the relativity of simultaneity is a bit hard to digest, and has been illustrated in Figure 4.1.3. The "spaceship-antispaceship annihilation" event is certainly a problem, but requires a more rigorous analysis than just drawing spacetime diagrams.

Fig. 4.2.1.: Light cone coordinates. λ selects a particular worldline, while τ selects a particular light cone.
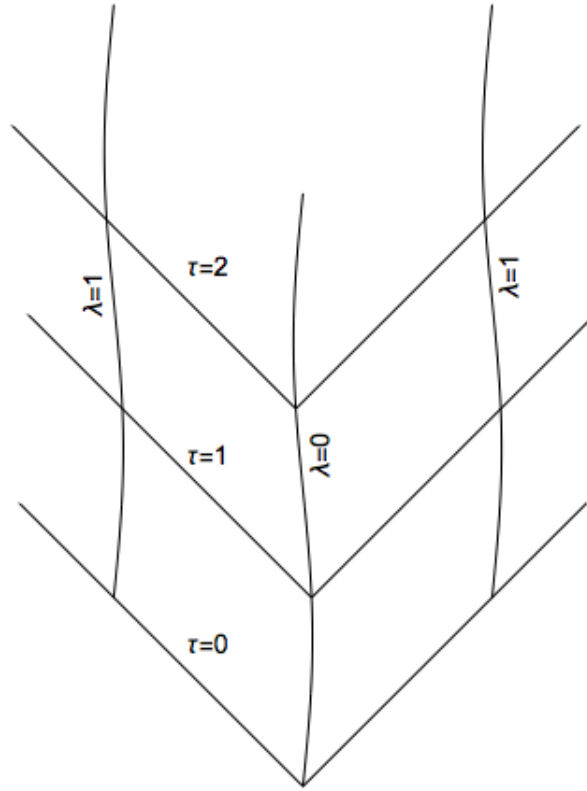
4.2   The 2+2 split

To introduce the 2+2 split, begin by considering a family of future-directed light cones emanating from a timelike worldline $x(\tau)$ where τ is proper time along the worldline. The boundary of the light cone consists of null geodesics with an affine parameter λ along each. Each null geodesic can be labeled by the proper time τ that corresponds to when it was emitted, thus defining a function which assigns to each point P the emission time τ of the null geodesic that passes through P This is illustrated in Figure 4.2.1 . The function τ corresponds to the retarded time.

At each point there is a null vector tangent to the null geodesics

$$k = \frac{\partial}{\partial \lambda} \tag{4.2.1}$$

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and a timelike vector

$$u = \frac{\partial}{\partial \tau} \quad . \tag{4.2.2}$$

The surfaces of constant λ and τ will be homeomorphic to the two sphere S^2 so that we can use spherical coordinates to locate points on them. These are the wavefront surfaces. We can then label the coordinates according to

$$x^0 = \tau, \qquad x^1 = \lambda, \qquad x^2 = \theta, \qquad x^3 = \phi \quad . \tag{4.2.3}$$

The corresponding basis vectors are

$$e_0 = u, \qquad e_1 = k, \qquad e_2 = \frac{\partial}{\partial \theta}, \qquad e_3 = \frac{\partial}{\partial \phi} \quad . \tag{4.2.4}$$

So, if we take a fixed value of τ, the fixed λ surfaces will foliate the light cone. Then the angular coordinates can be propagated from one constant λ surface to the next along the null geodesics. This effectively defines spherical wavefronts of constant λ, ϕ.

Since the vector field k is null, it satisfies the constraint

$$k \cdot k = 0 \quad , \tag{4.2.5}$$

Since λ is an affine parameter along the null geodesics it follows that we also have the constraint

$$D_k k = 0 \quad , \tag{4.2.6}$$

where $D_k k$ denotes the covariant derivative of k in the direction of k. Since λ is an arbitrary affine parameter it could differ up to a scaling factor b for a transformation like $\lambda \to b\lambda$.

Being timelike, the vector field u satisfies

$$u \cdot u = g_{00} = -f(\tau, \lambda) \quad , \tag{4.2.7}$$

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where $f(\tau, 0) = 1$ for any value of τ. Additionally k and u satisfy

$$[u, k] = 0 \quad . \tag{4.2.8}$$

If we wish to consider surfaces of constant θ, ϕ it is helpful to define two coordinate functions

$$t = \tau + \frac{\lambda}{\sqrt{2}} \quad , \tag{4.2.9}$$

and

$$r = \frac{\lambda}{\sqrt{2}} \quad . \tag{4.2.10}$$

In terms of this transformation, we will have

$$u = \frac{\partial}{\partial t} \quad , \tag{4.2.11}$$

and

$$k = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right) \quad . \tag{4.2.12}$$

We can think of the r and t coordinates as just ordinary distance and time. Thus, surfaces of constant θ, ϕ correspond to radial spacetime diagrams, where t would be the time an observer on a world line assigns to events by following the Einstein clock synchronization procedure. With the definitions of r and t we have effectively defined a Riemannian normal coordinate system, where $r = 0$ corresponds to flat space.

Using this fact and assuming zero torsion, we can actually narrow down some of the metric components by writing them as dot products. Indeed it can be shown that the metric components can at most be linear in λ and therefore some distinct components will vanish. The details of this calculation will be skipped for the purpose of conciseness, but it can be shown that the line element will take the form

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$$ds^2 = d\tau \left(-f d\tau - \sqrt{2} d\lambda + 2N_2 d\theta + 2N_3 d\phi \right) + g_{22} d\theta^2 + 2g_{23} d\theta d\phi + g_{33} d\phi^2 \quad . \quad (4.2.13)$$

We can next find an inward pointing null vector ℓ that generalizes the inward pointing one in Minkowski space. Being null this vector will satisfy $\ell \cdot \ell = 0$. Also we can assume normalization so that

$$k \cdot \ell = -1 \quad . \tag{4.2.14}$$

Now consider a vector

$$\hat{u} = \frac{1}{\sqrt{2}} (k + \ell) \quad . \tag{4.2.15}$$

This vector is timelike with magnitude 1 everywhere, so it cannot equal the vector u everywhere since u is only a unit vector at $\lambda = 0$. If we try a less restrictive assumption

$$\hat{u} = u + Bk \quad . \tag{4.2.16}$$

we can write

$$u + Bk = \frac{1}{\sqrt{2}} (k + \ell) \quad , \tag{4.2.17}$$

where B is a constant to be determined. Take the dot product of both sides of this equation with k to get

$$u \cdot k = \frac{1}{\sqrt{2}} (\ell \cdot k) = -\frac{1}{\sqrt{2}} \quad . \tag{4.2.18}$$

Then take the dot product with itself yielding

$$u \cdot u + 2Bu \cdot k + B^2 k \cdot k = \frac{1}{2} (k \cdot k + 2k \cdot \ell + \ell \cdot \ell) \quad . \tag{4.2.19}$$

These dot products are known, so that

$$-f + 2B \left(-\frac{1}{\sqrt{2}} \right) = -1 \quad , \tag{4.2.20}$$

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$$- f - \sqrt{2}B = -1 \quad , \tag{4.2.21}$$

or

$$B = \frac{1}{\sqrt{2}} \left(1 - f \right) \quad . \tag{4.2.22}$$

Substituting back in we find that

$$\hat{u} = u + \frac{1}{\sqrt{2}} \left(1 - f \right) k \quad , \tag{4.2.23}$$

which becomes u along the original worldline, where $f = 1$. We can now return to the equation

$$\hat{u} = \frac{1}{\sqrt{2}}(k + \ell) \quad . \tag{4.2.24}$$

Solving for ℓ yields

$$\ell = \sqrt{2}u - fk \quad . \tag{4.2.25}$$

In order to characterize the curvature of the spacetime, we want to define a projection tensor, however we will in fact need to use two projection tensors H and V. This is because we can project either onto the wavefront 2-surfaces (constant τ, λ) or onto the radial surfaces (constant θ, ϕ). For the wavefront projection tensor we can define

$$H = 1 + \ell \otimes g(k) + k \otimes g(\ell) \quad , \tag{4.2.26}$$

with symmetric components

$$H_{\mu\nu} = g_{\mu\nu} + \ell_\mu k_\nu + k_\mu \ell_\nu \quad . \tag{4.2.27}$$

The radial projection is just

$$V = 1 - H \quad , \tag{4.2.28}$$

with components

$$V_{\mu\nu} = -\ell_\mu k_\nu - k_\mu \ell_\nu \quad . \tag{4.2.29}$$

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With further calculations it can be shown that

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{32} & g_{33} \end{pmatrix} \quad , \tag{4.2.30}$$

and similarly

$$V = \begin{pmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{10} & g_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{4.2.31}$$

4.3 Further Goals: The Raychaudhuri Equations

This system of light cone coordinates is convenient in that it is consistent with our "warp tube" description. The only part of spacetime that isn't covered by these coordinates is the future domain of dependence $D^+(S)$, which can't be affected by anything we do, regardless. Unfortunately the calculations required to fully characterize the spacetime become increasingly complicated and have not yet been completed.

From the projection tensor definitions of the previous section, the plan of attack is to continue by defining the extrinsic curvature tensors, along with the other curvature objects to characterize the 2-surfaces. From that point the expansion, vorticity, and other elements can in principle be written out in order to set up the Raychaudhuri equations, which can tell us about how a bundle of geodesics will behave. The end goal is to judge whether geodesics must focus or defocus in the presence of some kind of matter. The way to use this to characterize superluminal travel is that if the spherical wavefront surfaces (cross sections of our light cones) are taken to be deforming then we could infer that the light cones are tilting or stretching in some way, which would be caused by the presence of some, presumably exotic, matter/energy distrubution.

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